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# LAMINAR BOUNDARY LAYER IN THE PRESENCE OF SUCTION

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*Institute of Hydromechanics, Academy of Sciences of the Ukrainian SSR*

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# LAMINAR BOUNDARY LAYER IN THE PRESENCE OF SUCTION

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## ANNOTATION

This monograph classifies the theoretical investigations of the influence of suction of a fluid through the permeable surface of a body on the characteristics of a laminar boundary layer and the aerodynamic drag. The case of an incompressible fluid is studied.

This monograph is intended for engineers, scientists, students and graduate students who are concerned with the technical applications of boundary layer theory in aviation, ship building and power machine construction. Editor-in-chief: V.M. Ivchenko.

# FOREWORD

The boundary layer theory was created more than 60 years ago. At the present time it is widely used by scientists and engineers in aviation, ship building and power machine construction. The well-known monographs of L.G. Loytsyanskiy [44] and G. Schlichting [55], in which basic attention was paid to investigations of the boundary layer on an impermeable surface, were devoted to a generalization of the boundary layer theory.

In recent times considerable attention has been paid to the development and refinement of particular problems of control of the boundary layer. In the present monograph an attempt has been made to systematically discuss the theory of control of the laminar boundary layer by means of suction of a fluid through the permeable surface of a body. One of the reasons for writing this monograph is the fact that almost all investigations on the theory of the laminar boundary layer in the presence of suction have been published in periodicals and have not been classified. In this monograph we look at the boundary layer of an incompressible fluid. The selection of materials for this monograph has been due to some degree to the scientific interest of the author.

The first chapter is devoted to deriving basic differential equations and obtaining the boundary conditions which take into account the specifics of this problem, i.e., the presence of suction of small amounts of fluid from the boundary layer throughout the permeable surface of a body. Furthermore in the same chapter the derivation is given for the integral relationships which have found wide use in the development of approximate methods for computing the boundary layer.

In the second chapter we discuss the available precise solutions to the equations of the laminar boundary layer in the presence of suction. The precise solutions are represented in the form of converging series or detailed tables obtained by using a computer. We studied especially the case which is of practical interest of the intensive suction of a laminar boundary layer.

In the theory of a laminar boundary layer any correctly formulated problem may be solved by numerical methods. However the qualitatively visible approximate analytical solutions are also important. In this respect Chapter 3 of the monograph is devoted to approximate methods of computing a plane boundary layer. Here the well-known approximate methods are given that are based on the integral relationship of impulses, energy and "three moments", and also the approximate methods of Shvets and Targ.

Chapter 4 is devoted to the problem of spatial boundary layer on a movable wing, and also on a body of revolution and of prolate bodies of arbitrary shape.

Chapter 5 is devoted to a problem that is of practical interest, i.e., a discussion of the existing methods of computing the laminar boundary layer with slotted suction. Here the methods of Wuest and Lachmann are studied. In conclusion of this chapter a simple approximate method of computation is given.

In the last, the sixth, chapter the basic problems of optimal suction of a laminar boundary layer are studied for the purpose of substantially reducing the resistance of bodies moving at high velocities. This problem is discussed not only from the position of the theory of hydrodynamic stability but also from the position of the conversion of the laminar boundary layer into a turbulent layer, taking into account the initial turbulence of the flow and the roughness of the surface of the body. Such an approach permits developing the methods of computing the optimal suction of a laminar boundary layer taking into account the basic factors encountered in technical applications.

It is not possible to cite a complete bibliography on the problem of suction of a laminar boundary layer. Therefore this book includes only a summary of the literature with which the author is sufficiently familiar and to which references are cited in the text.

Examples of the possible practical application of these questions have not been reflected in the monograph. To some degree the articles by the author [26,35] have been devoted to this purpose.

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# CHAPTER 1

## DIFFERENTIAL PRANDTL EQUATIONS. INTEGRAL RELATIONSHIPS AND BOUNDARY CONDITIONS.

### Deriving the Prandtl Equations. Basic Initial and Boundary Conditions

The steady isothermal flow of a viscous incompressible fluid /5* in the absence of mass forces is described by the Navier-Stokes equations which in a rectangular system of coordinates have the form [47]

$$\left.\begin{aligned}
\varrho\frac{\partial u}{\partial t}+\varrho u\frac{\partial u}{\partial x}+\varrho v\frac{\partial u}{\partial y}+\varrho w\frac{\partial u}{\partial z}&=-\frac{\partial p}{\partial x}+\mu\Delta u;\\
\varrho\frac{\partial v}{\partial t}+\varrho u\frac{\partial v}{\partial x}+\varrho v\frac{\partial v}{\partial y}+\varrho w\frac{\partial v}{\partial z}&=-\frac{\partial p}{\partial y}+\mu\Delta v;\\
\varrho\frac{\partial w}{\partial t}+\varrho u\frac{\partial w}{\partial x}+\varrho v\frac{\partial w}{\partial y}+\varrho w\frac{\partial w}{\partial z}&=-\frac{\partial p}{\partial z}+\mu\Delta w.
\end{aligned}\right\}\tag{1.1}$$

Here $t$ is the time; $u$, $v$ and $w$ are the projections of the velocity vector on the corresponding coordinate axes; $p$ is the pressure; $\rho$ and $\mu$ are the density and the dynamic coefficient of viscosity of the fluid; the operator $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$.

It is necessary to associate the equation of discontinuity with these equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.\tag{1.2}$$

Systems (1.1) and (1.2) consist of nonlinear differential equations in partial derivatives having a rather complex form. In final

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* Numbers in the margin indicate pagination in the foreign text.

form this system may be integrated only in a comparatively small number of particular cases. Development of the theory of motion of a viscous fluid has occurred mainly by the use of approximate methods of integrating the equations. /6

The equations of motion of a viscous fluid in mathematical form express the equilibrium of the forces of pressure and inertia as well as the forces of internal friction. If it is completely impossible to integrate the equations of motion of a viscous fluid, then in the process of developing approximate methods we can discard the value of one, i.e., the least important, of these forces. The significance of the forces of pressure can never be discarded since these are the internal forces by means of which the equilibrium of all the other forces is accomplished. If we ignore the forces of internal friction and take into account the forces of inertia we obtain the equation of the hydrodynamics of an ideal liquid. If we discard the forces of inertia and retain the forces of friction we arrive at equations of motion of a fluid with a high viscosity.

However, many fluids with the motion of which we are often concerned in solving practical problems have a very low viscosity. Such for example are water and air. If simultaneously the characteristic dimension of a streamlined body and a value of the rate of flow are high, we then obtain a flow with large Reynolds numbers. In such cases, as experiment shows, the influence of viscosity is expressed only near the surface of the streamlined body. This fact was noted as long ago as 1880 by D.I. Mendeleyev in his investigations which were devoted to the problem of resistance to the motion of liquids.

The influence of viscosity near the surface is substantial for the same reason that even liquids with an insignificant viscosity retain the property to adhere to a solid streamlined surface. In proportion to this distance from the surface the rate of motion of a fluid rapidly increases and at a certain distance $\delta$ from the surface assumes a value which is practically equal to the rate of external flow. This layer having a thickness $\delta$ in which the speed of the liquid near the surface of the body varies from zero to the speed of the external flow is called the boundary layer. The concept of a boundary layer was used by Zhukovskiy in reference [7] published in 1890. However, only Prandtl [89] in 1904 was able to formulate the basic differential equation of a fluid motion in the boundary layer. Prandtl proposed the mathematical theory of the boundary layer which was later found to be very fruitful and which received practical application in solving numerous problems introduced in such important fields of technology as airplane construction, ship building and power machine construction.

Let us derive equations of the laminar boundary layer for a steady flow on a plane surface. The axis $x$ is directed along the surface and the axis $y$ perpendicular to it. Using the above-mentioned assumption, the Navier-Stokes equations (1.1) and the /7 equation of discontinuity (1.2) can be significantly simplified and reduced to the form

$$\left.\begin{aligned} &\varrho u \frac{\partial u}{\partial x} + \varrho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu \Delta u; \\ &\varrho u \frac{\partial v}{\partial x} + \varrho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mu \Delta v; \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \end{aligned}\right| \qquad (1.3)$$

where the operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Let us assume that the boundary layer with a thickness $\delta$ is a layer in which the gradient of velocities $\partial u/\partial y$ is large and the longitudinal component of the velocity $u$ increases rapidly from zero on the surface of the speed of the external flow $U(x)$, determined from the solution to the respective equations of motion of an ideal fluid or experimentally. Let us also assume that the thickness of the boundary layer is much smaller than the characteristic dimension of the body $L(\delta << L)$.

Let $U_0$ be the chracteristic rate of the external flow. From the equation of discontinuity of the system (1.3) it follows that for a plane flow

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = 0\left(\frac{U_0}{L}\right). \qquad (1.4)$$

The designation 0 represents the order of magnitude. Term by term integration of expression (1.4) across the boundary layer from $y$ = 0 to $y = \delta$ yields the following equation

$$v = 0\left(\frac{U_0\delta}{L}\right).$$

Consequently, on the outer boundary of the boundary layer the lateral component of the velocity is a small value in comparison with the longitudinal component.

Let us compute the terms in the equations of system (1.3). In this case the derivative of the arbitrary function $F$ over $x$ is much smaller then the respective derivative of $y$. In fact,

$$\frac{\partial F}{\partial x} = 0\left(\frac{F}{L}\right); \quad \frac{\partial F}{\partial y} = 0\left(\frac{F}{\delta}\right).$$

Consequently the terms of the first and second equation of system (1.3) have the following order: /8

$$\varrho u \frac{\partial u}{\partial x},\ \varrho v \frac{\partial u}{\partial y},\ \frac{\partial p}{\partial x} \quad 0\left(\frac{\varrho U_0^2}{L}\right); \tag{1.5}$$

$$\mu \frac{\partial^2 u}{\partial y^2} \quad 0\left(\frac{\mu U_0}{\delta^2}\right); \tag{1.6}$$

$$\mu \frac{\partial^2 u}{\partial x^2} \quad 0\left(\frac{\mu U_0}{L^2}\right). \tag{1.7}$$

Let us note that the terms of equation (1.5) can be physically interpreted as the forces of inertia and terms of equation (1.6) as forces of viscosity per unit volume of the boundary layer. If we assume that the forces of viscosity in the boundary layer have the same order as the inertial forces we can estimate the thickness of the boundary layer

$$0\left(\frac{\varrho U_0^2}{L}\right) = 0\left(\frac{\mu U_0}{\delta^2}\right) \text{ or } \frac{\delta}{L} = 0\left(\mathrm{Re}^{-\frac{1}{2}}\right), \tag{1.8}$$

where $\mathrm{Re} = \frac{U_0 L}{\nu}$ is the Reynolds number.

In the first equation of system (1.3) if we discard the terms of the second order of smallness for the plane boundary layer with steady motion we find Prandtl's differential equation

$$\varrho u \frac{\partial u}{\partial x} + \varrho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}. \tag{1.9}$$

Similarly we can compute the terms of the second equation of system (1.3). It is obvious that they have the same order

$$\varrho u \frac{\partial v}{\partial x},\ \varrho v \frac{\partial v}{\partial y} \quad 0\left(\frac{\varrho U_0^2 \delta}{L^2}\right) = 0\left(\frac{\varrho U_0^2}{L} \cdot \frac{\delta}{L}\right); \tag{1.10}$$

$$\mu \frac{\partial^2 v}{\partial y^2} \quad 0\left(\frac{\mu U_0}{\delta^2} \cdot \frac{\delta}{L}\right); \tag{1.11}$$

$$\mu \frac{\partial^2 v}{\partial x^2} \quad 0\left(\frac{\mu U_0}{L^2} \cdot \frac{\delta}{L}\right). \tag{1.12}$$

In the second equation of system (1.3) if we discard terms of the second order of smallness, we obtain /9

$$\frac{\partial p}{\partial y} = 0\left(\frac{\varrho U_0^2}{L}\cdot\frac{\delta}{L}\right). \tag{1.13}$$

In other words the gradient of pressure $\partial p/\partial y$ is a small value and the change in the pressure across the boundary layer which has the order $0\left(\frac{\varrho U_0^2}{L}\cdot\frac{\delta}{L}\right)$ can be ignored. Then the second equation of system (1.3) is simplified and reduced to the form

$$p(x, y) = p(x). \tag{1.14}$$

Equation (1.14) has a trivial solution if in the Equation (1.9) $y$ tends to infinity:

$$\varrho U\frac{dU}{dx} = -\frac{dp}{dx}. \tag{1.15}$$

Consequently, in this case the pressure is fully determined by the rate of the external flow, its longitudinal derivative and the density of the fluid.

Analogously we can compute the terms of the third equation of system (1.1). If we also take into account that the time has the order $0(\frac{L}{U_0})$, the system of differential equations of a laminary spatial boundary layer and the equation of discontinuity with steady motion take the form

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\varrho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2};\\ &\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\varrho}\frac{\partial p}{\partial z} + \nu\frac{\partial^2 w}{\partial y^2};\\ &\frac{\partial p}{\partial y} = 0;\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \end{aligned}\right\} \tag{1.16}$$

This system of equations in the presence of suction of a fluid from the boundary layer must be solved with the following boundary conditions:

$$\begin{aligned} &u = w = 0;\ v = -v_0(x, t) \text{ with } y = 0;\\ &u \to U;\ v \to V;\ w \to W \text{ with } y \to \infty. \end{aligned} \tag{1.17}$$

Here $u_0(x,t)$ is the local rate of suction of the liquid from the boundary layer; $U$, $V$ and $W$ are the longitudinal, lateral and transverse components of the velocity at the outer boundary of the /10 boundary layer. In the case of steady motion, we must bear in mind the initial conditions: when $t = 0$ the function $u$ must be reduced to a given function of $x$ and $y$.

In the first two equations of system (1.16) the value of the pressure may be excluded by using the Bernoulli integral which is used on the outer boundary of the boundary layer:

$$p + \frac{1}{2}\varrho(U^2 + V^2 + W^2) = \text{const.} \tag{1.18}$$

## Supplementary Boundary Conditions on the Surface of a Body and the External Boundary of the Layer

There is no general method of solving the system of equations (1.16) with satisfaction of the initial and the boundary conditions (1.17). In recent times, considerable significance has been given to the approximate methods of computation based on the concept of the boundary layer of a finite thickness. In this case along with the basic boundary conditions (1.17) the supplementary boundary conditions are used for the velocities on the surface of a body and at the outer boundary of the boundary layer.

Let us explain the derivation of the above-mentioned boundary conditions. As it has already been noted [see the first boundary condition (1.17)] that for a boundary layer on the surface of a body (when $y = 0$) the conditions of adhesion ($u = w = 0$) and the presence of a normal component [rate of suction $v = -v_0(x)$] must be satisfied. From the concept of a layer of finite thickness it follows that at the external boundary the speed of the fluid in it (when $y = \delta$) is equal to the rate of the external flow ($u = U$). Let us also note that at the outer boundary of the boundary layer $v \neq 0$, i.e., the boundary of the boundary layer is not the line or the surface of the current. The requirement for smoothness in transition of the velocity in the boundary layer to the velocity of the external flow leads to satisfaction of the conditions when $y = \delta$:

$$u = U;\quad \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = \ldots = \frac{\partial^k u}{\partial y^k} = \ldots = 0 \;(k = 1, 2 \ldots \infty). \tag{1.19}$$

In fact no outer boundary of the boundary layer exists and the above boundary condition must be satisfied asymptotically, i.e., when $y \to \infty$.

Let us look at the derivation of the other boundary conditions on the surface of a body for a plane boundary layer. Assuming in equation (1.9) that $y = 0$, we obtain [if $u = 0$, $v = -v_0(x)$] /11

$$-v_0\left(\frac{\partial u}{\partial y}\right)_0 = -\frac{1}{\varrho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right)_0. \tag{1.20}$$

This condition connects the value of the second derivative of the velocity on the surface of a body with a gradient of pressures on the outer boundary of a boundary layer and the value of the rate of suction.

After differentiating again the left and right sides of this equation over $y$ we obtain

$$\frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial x}+u\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial y^2}=\nu\frac{\partial^3 u}{\partial y^3}. \qquad (1.21)$$

Let us note that from the equation of discontinuity of system (1.3) it follows that

$$\frac{\partial u}{\partial x}=-\frac{\partial v}{\partial y}. \qquad (1.22)$$

If we substitute equation (1.22) into equation (1.21) this latter equation is simplified and takes the form

$$u\frac{\partial^2 u}{\partial x\partial y}+v\frac{\partial^2 u}{\partial y^2}=\nu\frac{\partial^3 u}{\partial y^3}. \qquad (1.23)$$

On the surface of the body when $y = 0$ from equation (1.23) we find a new boundary condition

$$v_0\left(\frac{\partial^2 u}{\partial y^2}\right)_0=\nu\left(\frac{\partial^3 u}{\partial y^3}\right)_0 \qquad (1.24)$$

This condition connects the second and third derivatives of the arbitrary component of velocity in the boundary layer with the value of the velocity of suction on the surface of the body.

Analogously we can obtain an infinite set of boundary conditions both on the surface of the body and at the external boundary of the boundary layer for the components of the velocities $u$ and $v$. Below we cite the basic boundary conditions which were used in the existing approximate methods of computation:

$$\text{with } y=0 \quad u=w=0;\ v=-v_0(x)$$

$$UU'+\nu\left(\frac{\partial^2 u}{\partial y^2}\right)_0=-v_0(x)\left(\frac{\partial u}{\partial y}\right); \qquad (1.25)$$ /12

$$\nu\left(\frac{\partial^3 u}{\partial y^3}\right)_0=-v_0(x)\left(\frac{\partial^2 u}{\partial y^2}\right)_0;$$

$$\nu\left(\frac{\partial^2 v}{\partial y^3}\right)_0=-v_0(x)\left(\frac{\partial v}{\partial y}\right)_0;$$

$$\nu\left(\frac{\partial^3 v}{\partial y^3}\right)_0=-v_0\left(\frac{\partial^2 v}{\partial y^2}\right)_0;$$

with $y \to \delta$

$$u \to U;\ \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = \ldots = 0;$$

$$v \to V;\ \frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial y^2} = \ldots = 0. \tag{1.26}$$

The rate of suction $v_0(x)$ may be distributed across the entire surface or panels or concentrated in narrow diagonal slots. Figure 1 gives a schematic classification of the methods of organizing receiving organs of suction systems which exist in engineering technology.

With distributed suction of the liquid from the boundary layer through the permeable surface of the body (Fig. 1a) the rate of suction is continuous and not equal to the zero of the functions ($v_0 \neq 0$). With panel suction of the liquid from the boundary layer through the permeable panels (Fig. 1b) the rate of suction

$$v_0(x) \neq 0 \text{ with } x_1 < x \leqslant x_2;$$
$$v_0(x) = 0 \text{ with } x_1 > x \text{ и } x > x_2,$$

where $x_1$ and $x_2$ are the coordinates of the beginning and the end of the permeable panel. With suction of the liquid from the boundary layer through narrow diagonal slots (Fig. 1c), in the region of the slots the rate of suction is not equal to zero ($v_0 \neq 0$), and on the remaining nonpermeable surface, the normal velocity component (rate of suction) is identically equal to zero.

The classification used is quite arbitrary; however, it is useful in developing engineering methods for computing the characteristics of a boundary layer in the presence of suction of a fluid through the permeable surface of a body.

## Relationships Between Impulses and Energy for a Plane Boundary Layer. Generalized Integral Relationships

For an approximate computation of the characteristics of a laminar boundary layer on a porous surface in the presence of suction, methods have been developed which are based on the use of integral relationships obtained by conversion of the system of /13
Prandtl's differential equations.

Let us derive the equation of impulses. For a steady plane incompressible flow in a laminar boundary layer let us rewrite the the differential equation (1.9) and the equation of discontinuity in the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\varrho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}; \tag{1.27}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad (1.28)$$

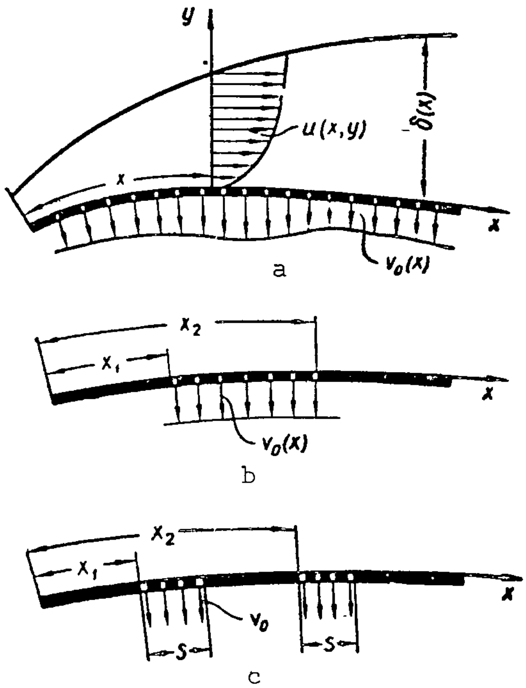



Fig. 1. Schematic of the Suction of a Boundary Layer.

Let us multiply both sides of equation (1.27) by $u$ and after term by term addition with equation (1.28) we obtain

$$2u\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}(uv) = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}. \qquad (1.29)$$

Here we take into account that from the Bernoulli equation applied to the flow on the external boundary of the boundary layer it follows that

$$-\frac{1}{\varrho}\cdot\frac{dp}{dx} = U\frac{dU}{dx}. \qquad (1.30)$$

Then we convert equation (1.29) to the form

$$-\frac{\partial}{\partial x}(U^2 - u^2) + U\frac{dU}{dx} + \frac{\partial}{\partial y}(uv) = \nu\frac{\partial^2 u}{\partial y^2} \qquad (1.31)$$

and term by term integration of $y$ from 0 to $\infty$ we obtain

$$-\int_0^\infty \frac{\partial}{\partial x}(U^2 - u^2)\,dy + \int_0^\infty U\frac{\partial U}{\partial x}\,dy + Uv_0 = -\frac{\tau_0}{\varrho}. \qquad (1.32)$$

With the ultimate derivation of equation (1.32) we have taken into account the boundary conditions

with $y = 0$ $u = 0$, $v = v_0$;
with $y = \infty$ $u = U$.

In these transformations we also use Newton's law /14

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0},$$

where $\tau_0$ is the local tangential stress on the surface of the body and the equation of discontinuity in the form

$$v = v_0 + \int_0^\infty \frac{\partial v}{\partial y} dy.$$ /15

Since, from the equation of discontinuity (1.28) it follows that $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$, then equation (1.32) can be written as

$$-\frac{d}{dx}[U^2(\delta^{**} + \delta^*)] + U\frac{d}{dx}[U\delta^*] + Uv_0 = -\frac{\tau_0}{\varrho}, \tag{1.33}$$

where $\delta^{**} = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{V}\right) dy$ is the thickness of loss of the impulse;

$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$ — is the thickness of the displacement.

After differentiation and transformation of equation (1.33) in final form we obtain

$$\frac{d\delta^{**}}{dx} + \frac{dU}{dx}\cdot\frac{\delta^{**}}{U}(H + 2) + \frac{v_0}{U} = \frac{\tau_0}{\varrho U^2}. \tag{1.34}$$

Equation (1.34) is the equation of impulses for the boundary layer. It may also be obtained by applying the law of momentum to the element of the boundary layer on a porous surface in the presence of suction.

In the particular case of a boundary layer on an impermeable surface ($v_0 \equiv 0$), equation (1.34) agrees with the integral relationship of Karman in the form in which it was first obtained by G. Holstein and T. Bohlen [75].

Let us proceed to deriving the equation of energy for the boundary layer. Adding the term $\frac{u}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ to the right-hand side of equation (1.27) and multiplying it by $u$ we obtain

$$\frac{3u^2}{2}\cdot\frac{\partial u}{\partial x} + \frac{1}{2}\cdot\frac{\partial}{\partial y}(u^2 v) - uU\frac{dU}{dx} = \nu u\frac{\partial^2 u}{\partial y^2}. \tag{1.35}$$

If we integrate equation (1.35) over $y$ from 0 to $\infty$, since /16

$$[u^2 v]_0^\infty = U^2\left[v_0 + \int_0^\infty \frac{\partial u}{\partial x} dy\right], \quad \left[u\frac{du}{\partial y}\right]_0^\infty = 0.$$

we obtain

$$\int_0^\infty \frac{3}{2} u^2 \frac{\partial u}{\partial x} dy + \frac{1}{2} U^2 \left[ v_0 - \int_0^\infty \frac{\partial u}{\partial x} dy \right] - \int_0^\infty \frac{u}{2} \frac{dU^2}{dx} = -\int_0^\infty \nu \left( \frac{\partial u}{\partial y} \right)^2 dy. \tag{1.36}$$

Let us convert equation (1.36) to the form

$$\frac{d}{dx}(U^3 \delta^{***}) = 2 \int_0^\infty \nu \left( \frac{\partial u}{\partial y} \right)^2 dy - v_0 U^2, \tag{1.37}$$

where $\delta^{***} = \int_0^\infty \frac{u}{U} \left[ 1 - \left( \frac{u}{U} \right)^2 \right] dy$ is the thickness of the energy loss.

The equation of energy (1.37) may also be obtained by applying the law of conservation of energy to the element of the boundary layer on a porous surface in the presence of suction. The equation of energy for the case of an impermeable surface ($v_0 \equiv 0$) was first obtained in 1935 by Leyvenzon [39]. Abroad the equation of energy became known only in 1948 after publication of the work of Weighardt [122].

In order to obtain more general integral relationships for the boundary layer [128], let us multiply equation (1.27) by $u^k$ and equation (1.28) by $\frac{u^{k+1}}{k+1}$, where $k = 0, 1, 2, \ldots$. If we add these expressions we obtain

$$\frac{\partial}{\partial x}(u^{k+2}) + \frac{\partial}{\partial y}(u^{k+1} v) = u^k (k+1) \left( U \frac{dU}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \right). \tag{1.38}$$

In deriving equation (1.38) it was taken into account that /17

$$-\frac{1}{\varrho} \cdot \frac{dp}{dx} = U \frac{dU}{dx};$$

$$\frac{\partial}{\partial x}(u^{k+2}) = (k+2) u^{k+1} \frac{\partial u}{\partial x};$$

$$\frac{\partial}{\partial y}(u^{k+1} v) = (k+1) v u^k \frac{\partial u}{\partial y} + u^{k+1} \frac{\partial v}{\partial y}.$$

Substituting equation (1.28) into expression (1.38) and term by term integration over $y$ from 0 to $\infty$ in general form we obtain the integral relationships for a plane boundary layer

$$\frac{1}{u^{k+1}}\frac{d}{dx}(u^{k+2}f)+g\frac{1}{U}\frac{dU}{dx}+\frac{v_0}{U}=h, \tag{1.39}$$

where

$$f=\int_0^\infty\frac{u}{U}\left[1-\left(\frac{u}{U}\right)^{k+1}\right]dy;$$

$$g=-(k+1)\int_0^\infty\frac{u}{U}\left[1-\left(\frac{u}{U}\right)^{k-1}\right]dy;$$

$$h=-(k+1)\int_0^\infty\left(\frac{u}{U}\right)^k\frac{\partial}{\partial y}\frac{\tau_0}{\varrho U^2}dy$$

are generalized integral thicknesses of the boundary layer.

If we assume in equation (1.39) that $k = 0$ and $k = 1$ we obtain respectively the equations of impulses and energy. For $k > 1$ this equation has no obvious physical interpretation.

For the particular case of an impermeable surface ($v_0 = 0$) the generalized integral relationship (1.39) was obtained by Golubev and first published in monograph [47].

## Relationship of Impulses and Energy for a Spatial Boundary Layer /18

The relationship of impulses for a spatial boundary layer can be derived from the physical arguments using the law of the conservation of the momentum for the element of the spatial boundary layer on a porous surface in the presence of suction [33].

Let us look at the element of spatial boundary layer on the porous surface of a body (Fig. 2) using a natural curvilinear system of coordinates. The $x$-axis is directed along the line of the current of the potential streamlining of the body, the $y$-axis perpendicular to the surface of the body and the $z$-axis along the line orthogonal to the line of the current in the plane adjacent to the surface of the body.

Let us choose on an arbitrary line of the current a certain point 1 which is found at a distance $x$ from the critical and let us draw through it two normals to the line of the current. The first normal $z$ is drawn in a plane tangent to the surface of the body up to intersection with the neighboring line of the current, and the second perpendicular to the tangential plane up to intersection with the outer boundary of the boundary layer. Let us denote this branch cut of the first normals by $\Delta z$ and the branch cut of the second (thickness of the boundary layer) as $\delta$. At the point of intersection of the normal $z$ with a neighboring line of current at a distance $\Delta z$ (point 2) let us establish the second normal to the surface of the body up to intersection with the outer boundary of the boundary layer. This branch cut will differ from $\delta$ by an infinitely small value. Let us give the increment to coordinate $x$ equal to $dx$, and at points 5 and 6 let us make the same geometric constructions as at point 1.

As a result of such a plotting we obtain a curvilinear unit volume bounded by the permeable surface of the body 1265, for the surface of the outer boundary of the boundary layer 4378, by two surfaces of the current 1485 and 2376 and by the two surfaces 1234 and 5678 (cross sections A and B), normal to the surface of the current.

Let us supply to the constructed curvilinear element of the spatial boundary layer the general law of variation and momentum similar to that done by Karman [80] for the particular case of a plane boundary layer forming on the profile with an impermeable surface.

From the law of momentum it follows that variation of the momentum of a fluid per given volume is equal to the impulse of all external forces applied to this fluid. For steady flow per interval of time $dt$ the change in the momentum of this fluid occupying /19
at the beginning of this segment of time the spatial volume 12345678, is composed of two values.

(1) For this period of time a certain amount of the fluid comprising the volume at the beginning of the interval flows from the limits of this volume. The momentum of the particles of this part of the fluid will be assumed to be a positive value. Then the corresponding value for the particles of the fluid flowing for the time $dt$ into the volume must be assumed to be negative and equal to the momentum carried through the walls of the spatial element.

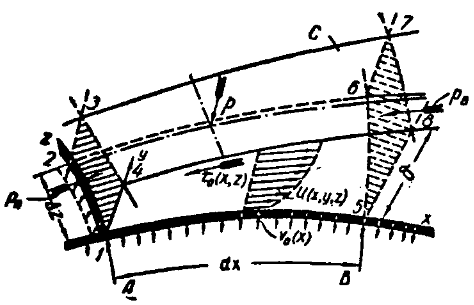


Fig. 2. Element of the Spatial Boundary Layer on a Porous Surface in the Presence of Suction.

For the interval of time $dt$ through the plane $A$ (see Fig. 2) the momentum of the liquid is carried which is equal to $-dxdt\int_0^\delta \varrho u^2 \Delta z dy$.

and through the plane $B$ equal to $\left(dt\int_0^\delta \varrho u^2\Delta z dy\right)$. Finally, through the planes $A$ and $B$ the momentum of the fluid is carried which is equal to

$$dxdt\frac{\partial}{\partial x}\int_0^\delta \varrho u^2\Delta z dy. \tag{1.40}$$

It is also obvious that through the planes $A$ and $B$ an amount of fluid flows /20

$$dxdt\frac{\partial}{\partial x}\int_0^\delta \varrho u\Delta z dy. \tag{1.41}$$

(2) As a result of the fact that the surface of this body is principle the amount of fluid flowing through its permeable surface 1256 for the same period of time is equal to

$$\varrho v_0\Delta z dx dt, \tag{1.42}$$

where $v_0(x,z)$ is a local velocity of suction of the liquid through the permeable surface of the body.

The total amount of fluid flowing from this element of a spatial boundary layer is equal to

$$dxdt\frac{\partial}{\partial x}\int_0^\delta \varrho u\Delta z dy + dxdt\varrho v_0\Delta z. \tag{1.43}$$

From the condition of discontinuity it follows that the same amount of fluid determined by formula (1.43) must flow through the surface 3487. It is obvious that on this surface the Equation $u = U$ is valid since at the outer boundary of the boundary layer when $y = \delta\ (x,z)$ the velocity $u$ is converted into the velocity of the external potential flow. Consequently, through the surface 3487 momentum flows that is equal to

$$-\varrho U\left(\frac{\partial}{\partial x}\int_0^\delta u\Delta z dy dx dt + v_0\Delta z dx dt\right). \tag{1.44}$$

and the total change in the momentum in this fluid corresponds to

$$\varrho\left(\frac{\partial}{\partial x}\int_0^\delta u^2\Delta z dy - U\frac{\partial}{\partial x}\int_0^\delta u\Delta z dy + v_0 U\Delta z\right)dxdt. \tag{1.45}$$

Now let us proceed to a computation of the impulse of the external forces. Since the force of friction, the value of which per unit of area is equal to $\tau_0$, operates on the plane 1265, the impulse of the forces of friction is equal to $-\tau_0 dx\Delta z dt$. At the same time /21 the sum of the projections on the $x$-axis of the forces of pressure applied to the planes $A$, $B$ and $C$ is equal to $-\delta\frac{\partial p}{\partial x}dx\Delta z$. Then for the impulse of these forces we can write

$$-\tau_0 dx\Delta z dt - \delta\frac{\partial p}{\partial x}dx\Delta z dt. \tag{1.46}$$

If we equate the change in the momentum (1.45) computed earlier to the total impulse of all forces (1.46) we find the integral relationship of impulses for the spatial boundary layer:

$$\frac{d}{dx}\int_0^\delta u^2\Delta z(y)\,dy - U\frac{d}{dx}\int_0^\delta u\Delta z(y)\,dy + \frac{v_0}{U}\Delta z = -\frac{1}{\varrho}\frac{dp}{dx}\Delta z(y)\,\delta - \frac{\tau_0}{\varrho}\Delta z(y). \tag{1.47}$$

Let us assume that the thickness of the boundary layer at a given point is negligibly small in comparison with the radius of the lateral curvature of the surface of the body. In this case we may assume that $\Delta z$ does not depend on the coordinate $y$. Then the left-hand side of equation (1.47) can be converted to the form

$$\Delta z\frac{d}{dx}\int_0^\delta u^2 dy - \Delta z\frac{d}{dx}\int_0^\delta u dy + U^2\frac{d\Delta z}{dx} + \frac{v_0}{U}\Delta z. \tag{1.48}$$

If we use the Bernoulli theorem for the external potential flow we obtain

$$-\frac{1}{\varrho}\cdot\frac{dp}{dx} = U\frac{dU}{dx}. \tag{1.49}$$

In this case equation (1.47) can be somewhat simplified and reduced to the form

$$\frac{d}{dx}\int_0^\delta u^2 dy - U\frac{d}{dx}\int_0^\delta u dy + U^2\frac{d\Delta z}{dx}\cdot\frac{1}{\Delta z} + \frac{v_0}{U} = U\frac{dU}{dx}\delta - \frac{\tau_0}{\varrho}. \tag{1.50}$$

Let us look at particular cases of equation (1.50).

For the axisymmetric boundary layer on a body of revolution with a porous surface in the presence of suction /22

$$\Delta z = r_0(x)\Delta\theta,$$

where $r_0(x)$ is the instantaneous radius of the body of revolution and $\Delta\theta$ = const is the angle between the two planes which pass through the axis of symmetry of the body of revolution.

In this case the integral relationship of impulses (1.50) can be simplified and reduced to the form [17]:

$$\frac{d}{dx}\int_0^\delta u^2dy - U\frac{d}{dx}\int_0^\delta udy + U^2\frac{dr_0}{dx}\cdot\frac{1}{r_0} + \frac{v_0}{U} = U\frac{dU}{dx}\delta - \frac{\tau_0}{\varrho}. \qquad (1.51)$$

For a plane boundary layer

$$\Delta z = \text{const}$$

and the integral relationship of the impulses (1.50) is reduced to the form

$$\frac{d}{dx}\int_0^\delta u^2dy - U\frac{d}{dx}\int_0^\delta udy + \frac{v_0}{U} = U\frac{dU}{dx}\delta - \frac{\tau_0}{\varrho}, \qquad (1.52)$$

obtained by Prandtl [46].

Just as the relationship of impulses, the relationship of energy for the spatial boundary layer can be derived from the physical argument using the law of the conservation of energy for the spatial boundary layer on a porous surface in the presence of suction [36].

Let us apply to the volume of the liquid included in the curvilinear element of the spatial boundary layer (see Fig. 2) the general law of the conservation of energy, taking into account suction of the fluid through the porous surface of the body. Let us look at the case of a steady flow and an incompressible fluid. If we compute the kinetic energy of the fluid for this element of the boundary layer, we obtain

$$T = dx\int_0^\delta \frac{1}{2}\varrho u^2\Delta z dy. \qquad (1.53)$$

In deriving expression (1.53) it was assumed that the lateral component of the velocity in the boundary layer is negligibly small in comparison with the longitudinal component.

For a certain interval of time $dt$ the change in the kinetic energy of the fluid occupying at the beginning of this interval of time the spatial volume 12345678 consists of three terms: the kinetic /23

energy $dt \int_0^\delta \frac{1}{2} \rho\, u^2 u \Delta z dy$, flowing through plane $B$; the kinetic energy $dt \int_0^\delta \frac{1}{2} \rho u^2 u \Delta z dy$, flowing through plane $A$ and the kinetic energy flowing through the surface 3478, $dtdx \frac{U^2}{2} \int_0^\delta \rho u \Delta z dy$. This latter expression is valid since for this surface, it being an external boundary of the boundary layer, the velocity $u$ converts to the velocity of the external potential flow $U$.

Since the surface 1265 of the body is permeable, then the amount of fluid flowing for the time $dt$ through the surface is equal to

$$\varrho v_0 \Delta z dx dt,$$

where $v_0(x,z)$ is the local rate of suction of the fluid through the permeable surface. For the kinetic energy of the sucked-off fluid we obtain

$$\frac{1}{2} \varrho U^2 v_0 \Delta z dx dt.$$

The total change in the kinetic energy of this system is

$$dT = dxdt\left(\frac{d}{dx}\int_0^\delta \frac{1}{2}\varrho u^3 \Delta z dy - \frac{U^2}{2}\frac{d}{dx}\int_0^\delta \varrho u \Delta z dy + \frac{1}{2}\varrho v_0 U^2 \Delta z\right). \qquad (1.54)$$

Let us proceed to computation of the works of external forces on the control surface of the spatial element of the boundary layer. The operation of the forces of viscosity on the surface of the body is equal to zero as a result of the adhesion of the fluid to the surface. On the other parts of the control surface we can ignore the operation of the forces of viscosity.

Operation of the forces of pressure on the surfaces of the element of the boundary layer for the time $dt$ is expressed by the integral

$$dA = -\iint_S p v_n d\sigma dt, \qquad (1.55)$$

where $v_n$ is a projection of the velocity on the outer normal to the control surface; $p$ is the pressure; $S$ is the control surface. /24

Using the well-known formula of Ostrogradskiy-Gauss, the surface integral (1.55) can be converted into a volume integral

$$\iint_S \rho v_n d\sigma = \iiint_V \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right) dV, \qquad (1.56)$$

where $v$ and $w$ are the projections of velocity in the boundary layer respectively on the axes $y$ and $z$; $V$ is the volume of the element of the boundary layer bounded by the control surface $S$.

Furthermore,

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \frac{\partial p}{\partial x}u + \frac{\partial p}{\partial y}v + \frac{\partial p}{\partial z}w. \qquad (1.57)$$

Since the fluid is incompressible, the first term in expression (1.57) is then equal to zero. Along with this on the differential equations of Prandtl we know that in the region of the boundary layer $\frac{\partial p}{\partial y} = 0$ (1.16). The selected system of coordinates directed along the lines of the current permit setting $w = 0$. Therefore the work of the forces of pressure

$$dA = -dt \iiint_V \frac{\partial p}{\partial x} u dV = -dt dx \frac{\partial p}{\partial x} \int_0^\delta u \Delta z dy. \qquad (1.58)$$

The energy of the forces applied to the particles of the viscous fluid diverge by the increase of kinetic energy of these particles and is partially dissipated, being converted into heat energy. If we ignore the small terms of higher order for the function of dissipation [47], we obtain

$$E = \mu\left(\frac{\partial u}{\partial y}\right)^2. \qquad (1.59)$$

Then for the time $dt$ and this volume of liquid the dissipating amount of energy

$$dB = dt \iiint_V \mu\left(\frac{\partial u}{\partial y}\right)^2 dV = dt dx \int_0^\delta \mu\left(\frac{\partial u}{\partial y}\right)^2 \Delta z dy. \qquad (1.60)$$

According to the law of the conservation of energy /25

$$dT = dA - dB, \tag{1.61}$$

and assuming that $\Delta z$ does not depend on the coordinate $y$, and taking into account expressions (1.54), (1.58) and (1.60) we obtain

$$\begin{gathered} dtdx \frac{d}{dx} \Delta z \int_0^\delta \frac{1}{2} \varrho u^3 dy - dtdx \frac{1}{2} U^2 \frac{d}{dx} \Delta z \int_0^\delta \varrho u dy + \\ + \frac{1}{2} \varrho v_0 U^2 \Delta z dx dt = - dtdx \Delta z \frac{dp}{dx} \int_0^\delta u dy - dtdx \Delta z \int_0^\delta \mu \left( \frac{\partial u}{\partial y} \right)^2 dy. \end{gathered} \tag{1.62}$$

By differentiating the first two terms of equation (1.62) and after dividing both sides of this equation by $dtdx\Delta z$ we find

$$\begin{gathered} \frac{d}{dx} \int_0^\delta \frac{1}{2} \varrho u^3 dy - \frac{U^2}{2} \frac{d}{dx} \int_0^\delta \varrho u dy + \frac{1}{2} \varrho v_0 U^2 + \\ + \frac{1}{\Delta z} \frac{d\Delta z}{dx} \left( \int_0^\delta \frac{1}{2} \varrho u^3 dy - \frac{U^2}{2} \int_0^\delta \varrho u dy \right) = - \frac{dp}{dx} \int_0^\delta u dy - \\ - \mu \int_0^\delta \left( \frac{\partial u}{\partial y} \right)^2 dy. \end{gathered} \tag{1.63}$$

For the particular case of a plane boundary layer on an impermeable surface of a body, i.e., in the case when $\Delta z$ = const and $v_0 \equiv 0$, the integral relationship of the energy (1.63) for the boundary layer was first obtained by Leyvenzon [39] in 1935. Apparently in spite of this in 1948 this relationship was obtained abroad by Weighardt [122], in the following form:

$$\int_0^\delta \frac{1}{2} \varrho u^3 dy - \frac{U^2}{2} \int_0^\delta \varrho u dy = - \frac{1}{2} \varrho U^3 \delta^{***}, \tag{1.64}$$

where $\delta^{***} = \int_0^\delta \frac{u}{U} \left[ 1 - \left( \frac{u}{U} \right)^2 \right] dy$ is the thickness of the energy loss.

Using expressions (1.15) and (1.64) let us derive the integral relationship of energy (1.63) for the spatial boundary layer on a porous surface of a body in the presence of suction to a final form: /26

$$\frac{d}{dx}(U^3\delta^{***}) - \frac{1}{\Delta z}\frac{d\Delta z}{dx}(U^3\delta^{***}) = 2\nu\int_0^\delta\left(\frac{\partial u}{\partial y}\right)^2 dy + v_0U^3, \tag{1.65}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic coefficient of viscosity of the fluid.

Let us look at the particular case of equation (1.65).

For the axisymmetric boundary layer on a body of revolution with a porous surface in the presence of suction

$$\Delta z = r_0(x)\Delta\theta.$$

In this case the integral energy relationship (1.65) is simplified and reduced to the form

$$\frac{d}{dx}(U^3\delta^{***}) - \frac{1}{r_0}\frac{dr_0}{dx}(U^3\delta^{***}) = 2\nu\int_0^\delta\left(\frac{\partial u}{\partial y}\right)^2 dy + v_0U^3. \tag{1.66}$$

For an impermeable surface ($v_0 \equiv 0$) equation (1.66) was obtained by Truckenbrodt [116].

For a plane boundary layer

$$\Delta z = \text{const}$$

and the integral energy relationship (1.65) is reduced to the form

$$\frac{d}{dx}(U^3\delta^{***}) = 2\nu\int_0^\delta\left(\frac{\partial u}{\partial y}\right)^2 dy + v_0U^3, \tag{1.67}$$

obtained by Head [73].

## Integral Relationships of Moments

L.G. Loytsyanskiy [43] showed that, by using a system of integral relationship of moments of the basic differential equation of the boundary layer and using simple families of velocity profiles, it is possible to develop a satisfactory method for integrating equations of the laminar boundary layer on an impermeable surface. Let us derive the integral relationships of moment for the equations of a plane laminar boundary layer on a porous surface in the presence of suction [13]. /27

The system of these equations and the basic boundary conditions for a steady flow in an incompressible fluid in the absence of volume forces has the form of (1.9) and (1.17). Using the equation of discontinuity (1.3) equation (1.9) may be given a form which is more suitable for further computation:

$$\frac{d}{dx}[u(U-u)]+\frac{\partial}{\partial y}[v(U-u)]+U'(U-u)-\nu\frac{\partial^2(U-u)}{\partial y^2}=0. \tag{1.68}$$

Let us multiply both sides of the equation (1.68) by $y^k$, where $k = 0, 1, 2, 3...$, and let us integrate over $y$ from 0 to $\infty$. As a result of the computations we obtain

$$\begin{aligned}\frac{d}{dx}\int_0^\infty y^k u(U-u)\,dy+\int_0^\infty y^k\frac{\partial}{\partial y}[v(U-u)]\,dy+\\ +U'\int_0^\infty y^k(U-u)\,dy-\nu\int_0^\infty y^k\frac{\partial^2(U-u)}{\partial y^2}\,dy=0.\end{aligned} \tag{1.69}$$

Equation (1.69) is an integral relationship of moments. In the future we will assume that the integrals in equation (1.69) will have finite values.

Let us look at the particular case of equation (1.69) when $k = 0$:

$$\frac{d}{dx}\int_0^\infty u(U-u)\,dy+U'\int_0^\infty(U-u)\,dy+\int_0^\infty\frac{\partial}{\partial y}[v(U-u)]dy=\frac{\tau_0}{\varrho}. \tag{1.70}$$

Here the quantity $\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is the intensity of the friction on the surface of the streamlined body, since,

$$\int_0^\infty\frac{\partial}{\partial y}[v(U-u)]\,dy=v_0U. \tag{1.71}$$

Equation (1.70) which represents the equation of impulse can be transformed to the form /28

$$\frac{d\delta^{**}}{dx}+\frac{U'\delta^{**}}{U}(2+H)+\frac{v_0}{U}=\frac{\tau_0}{\varrho}. \tag{1.72}$$

Here $\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$ is the thickness of the displacement; $\delta^{**} = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$ is the thickness of the impulse loss; $H = \frac{\delta^*}{\delta^{**}}$ is the form parameter of the boundary layer.

By introducing into study the dimensionless parameters

$$f = \frac{\delta^{**2}}{\nu} \cdot \frac{dU}{dx}; \quad \zeta = \frac{\tau_0 \delta^{**}}{\mu U}; \quad t^{**} = \frac{v_0 \delta^{**}}{\nu} \tag{1.73}$$

and deriving the algebraic transformation, we can reduce equation (1.72) to the form

$$\frac{df}{dx} = \frac{U''}{U'} f + \frac{U'}{U} \{2\zeta - 2\,[2 + H]\,f - 2t^{**}\}, \tag{1.74}$$

where

$$U'' = \frac{\partial^2 U}{\partial x^2}.$$

In the future equation (1.74) will be called an integral relationship of impulses or the equation of zero moment.

When $k = 1$, from equation (1.69) we find the equation of the first moment. In this case we can write

$$\frac{d}{dx} \int_0^\infty yu\,(U - u)\,dy + \int_0^\infty y \frac{\partial}{\partial y}\,[v\,(U - u)]\,dy - U' \int_0^\infty y\,(U - u)\,dy =$$
$$= \nu \int_0^\infty \frac{\partial^2 (U - u)}{\partial y^2}\,dy. \tag{1.75}$$

If we compute the second and fourth integrals of equation (1.75), we obtain

$$\int_0^\infty y \frac{\partial}{\partial y}\,[v\,(U - u)]\,dy = - \int_0^\infty v\,(U - u)\,dy; \tag{1.76}$$ /29

$$\int_0^\infty y \frac{\partial^2 (U - u)}{\partial y^2}\,dy = U. \tag{1.77}$$

Taking formulas (1.76) and (1.77) into account, we can write equation (1.75) in the form

$$\frac{d}{dx}\int_0^\infty yu\,(U-u)\,dy-\int_0^\infty v\,(U-u)\,dy+U'\int_0^\infty y\,(U-u)\,dy=\nu U. \tag{1.78}$$

The first integral of equation (1.78)

$$\int_0^\infty yu\,(U-u)\,dy=H_1U^2\delta^{**2}, \tag{1.79}$$

where

$$H_1=\int_0^\infty \eta\varphi\,(1-\varphi)\,d\eta;\quad \eta=\frac{y}{\delta^{**}};\quad \frac{u}{U}=\varphi.$$

Noting that

$$\frac{d\eta}{dx}=-\frac{\eta}{\delta^{**}}\cdot\frac{d\delta^{**}}{dx}, \tag{1.80}$$

from the equation of discontinuity and the boundary condition of the surface of the body we have

$$v=-\int_0^y\frac{\partial u}{\partial x}\,dy-v_0=U'\delta^{**}\int_0^\eta\varphi\,d\eta-U\frac{d\delta^{**}}{dx}\left(\int_0^\eta\varphi\,d\eta-\eta\varphi\right)-v_0\,. \tag{1.81}$$

The second integral from equation (1.78) taking formula (1.81) into account can be written as

$$\int_0^\infty v(U-u)\,dy=H_2U^2\delta^{**}\frac{d\delta^{**}}{dx}-H_3UU'\delta^{**2}-v_0\int_0^\infty(U-u)\,dy=$$
$$=H_2U^2\delta^{**}\frac{d\delta^{**}}{dx}-H_3UU'\delta^{**2}-v_0U\delta^*, \tag{1.82}$$ /30

where

$$H_2=\int_0^\infty\left(\eta\varphi-\int_0^\eta\varphi\,d\eta\right)(1-\varphi)\,d\eta;$$

$$H_3=\int_0^\infty\left(\int_0^\eta\varphi\,d\eta\right)(1-\varphi)\,d\eta.$$

Finally the third integral in the equation is converted to the form

$$\int_0^\infty y(U-u)\,dy = H_4 U\delta^{**2}, \tag{1.83}$$

where

$$H_4 = \int_0^\infty \eta(1-\varphi)\,d\eta.$$

If we substitute integrals (1.79), (1.82) and (1.83) into equation (1.78) after several transformations we obtain

$$\begin{gathered}\frac{d}{dx}\left(H_1 U^2\delta^{**2}\right) - H_2 U^2\delta^{**}\frac{d\delta^{**}}{dx} + H_3 UU'\delta^{**2} + v_0 U\delta^{*} + \\ + H_4 UU'\delta^{**2} = \nu U.\end{gathered} \tag{1.84}$$

If we substitute $\frac{\delta^{**2}}{\nu} = \frac{f}{U'}$ into equation (1.84) we find

$$\frac{df}{dx} = \frac{1}{\left(H_1 - \frac{1}{2}H_2\right)}\cdot\frac{U'}{U}\left[1 - Ht^{**} - (2H_1 + H_3 + H_4)f\right] + \frac{U''}{U'}f. \tag{1.85}$$

When $k = 2$ from equation (1.69) we obtain the equation of the second moment. In this case

$$\begin{gathered}\frac{d}{dx}\int_0^\infty y^2 u(U-u)\,dy - 2\int_0^\infty yv(U-u)dy + U'\int_0^\infty y^2(U-u)dy = \\ = 2\nu\int_0^\infty (U-u)\,dy.\end{gathered} \tag{1.86}$$

It is obvious that /31

$$\int_0^\infty y^2 u(U-u)\,dy = H_5 U^2\delta^{**3}; \tag{1.87}$$

$$\int_0^\infty yv(U-u)\,dy = H_6 U^2\delta^{**2}\frac{d\delta^{**}}{dx} - H_7 UU'\delta^{**3} - H_4 v_0\delta^{**2}U; \tag{1.88}$$

$$\int_0^\infty (U-u)\,dy = U\delta^{**}H, \tag{1.89}$$

where

$$H_5 = \int_0^\infty \eta^2\varphi(1-\varphi)\,d\eta;$$

$$H_6 = \int_0^\infty \eta\left(\eta\varphi - \int_0^\eta \varphi d\eta\right)(1-\varphi)\,d\eta;$$

$$H_7 = \int_0^\infty \eta\left(\int_0^\eta \varphi d\eta\right)(1-\varphi)\,d\eta.$$

Then, taking expressions (1.87)-(1.89) into account after several transformations of formula (1.86) we find the equation of the second moment in the final form

$$\frac{df}{dx} = \frac{U'}{U}\frac{4}{(3H_5 - 2H_6)}\left[H - H_4 t^{**} - \left(H_5 + H_7 + \frac{1}{2}H_8\right)f\right] + \frac{U''}{U'}f, \tag{1.90}$$

where

$$H_8 = \int_0^\infty \eta^2(1-\varphi)\,d\eta.$$

# CHAPTER 2

## CERTAIN PRECISE SOLUTIONS OF THE PRANDTL EQUATION

### The Boundary Layer on a Plate

For the particular case of longitudinal streamlining of a flat plate, when the longitudinal pressure gradient is equal to zero, i. e., $\frac{\partial p}{\partial x} = 0$, the differential equation for the boundary layer (1.9) and the third equation of system (1.3), the equation of continuity, can be written in the form /32

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\frac{\partial^2 u}{\partial y^2};\\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \end{aligned}\right\} \quad (2.1)$$

with the boundary conditions

$$\begin{aligned} u = 0,\quad v = -v_0(x) \text{ with } y = 0;\\ u \to U_0 \text{ with } y \to \infty. \end{aligned} \quad (2.2)$$

A simple solution of system (2.1) in its final form can be obtained with uniform suction of fluid from the boundary layer ($v_0$ = const). In this case system (2.1) has a solution first obtained by Griffith and Meredith [70] for which the velocity distribution in different sections of the boundary layer is not a function of the current coordinate $x$. Since in this case $\frac{\partial u}{\partial x} = 0$, then it follows from the equation of continuity of system (2.1) that

$$v(x, y) = v_0 = \text{const} < 0.$$

Therefore the first equation of system (2.1) has the form /33

$$v_0\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2},$$

whose solution will be

$$u(y) = U_0\left[1 - \exp\left(\frac{v_0 y}{\nu}\right)\right]. \qquad (2.3)$$

Figure 3 shows the velocity distribution across the boundary layer for an asymptotic profile (2.3) with suction and a comparison

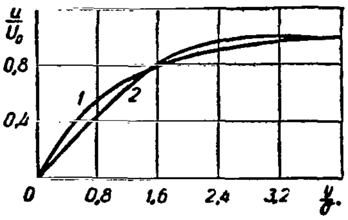



Fig. 3. Velocity Distribution Across the Boundary Layer of a Longitudinally Streamlined Plate: (1) Is the Asymptotic Profile with Uniform Suction; (2) Is the Velocity Profile Without Suction (According to Blasius).

of the asymptotic profile with the Blasius profile without suction. The solution obtained in final form (2.3) is also a precise solution of the entire system of Navier-Stokes equations (1.3).

The displacement thickness, the thickness of the impulse loss and the tangential stress on the surface can be determined in this case by simple calculations allowing for expression (2.3):

$$\left.\begin{aligned} \delta^* &= \int_0^\infty \left(1 - \frac{u}{U_0}\right) dy = \frac{\nu}{-v_0}; \\ \delta^{**} &= \int_0^\infty \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) dy = \frac{1}{2}\frac{\nu}{-v_0}; \\ \tau_0 &= \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \varrho(-v_0 U_0). \end{aligned}\right\} \qquad (2.4)$$

The tangential stress is a function only of the density of the fluid and is not a function of its viscosity.

The relationships thus obtained are valid only for longitudinal /34
streamlining of a flat plate with uniform suction beginning at a certain distance from the leading edge. R. Iglisch [78] pointed out that, on a plate, the boundary layer reaches an asympotic state after passing the initial section $x_p$, determined by the inequality

$$x_p > \frac{4\nu U_0}{(-v_0)^2}. \qquad (2.5)$$

The velocity profiles in the initial part, which Iglisch obtained by numerical integration of system (2.1) (Figure 4), were not affine to one another. Near the leading edge ($\xi$ = 0.1), the velocity profile has a shape similar to the shape of the Blasius profile without suction, and in proportion to the increase in the parameter $\xi$ = $\left(\frac{-v_0}{U_0}\right)\frac{\sqrt{2U_0x}}{\nu}$ approaches the asymptotic.

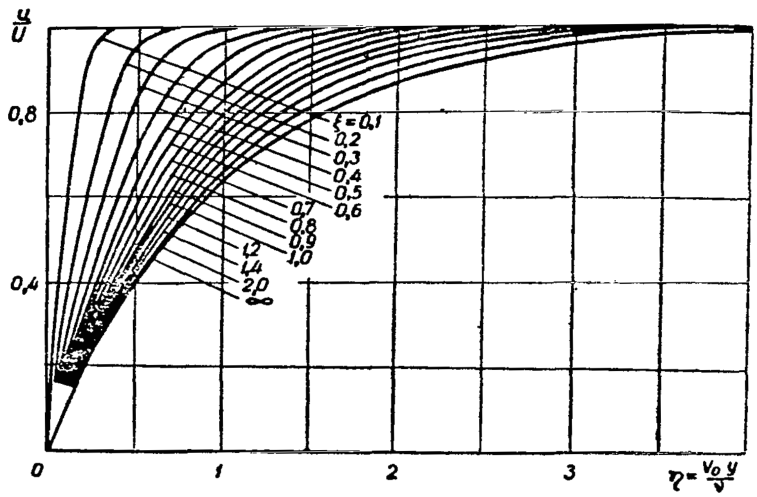



Fig. 4. Distribution of Velocities Across the Boundary Layer in the Accelerating Part of a Flat Plate With Uniform Suction

Table 1 gives the data about the change in the displacement thickness, the thickness of the impulse loss and the tangential stress in the initial section of the boundary layer with uniform suction as a function of the parameter $\xi$.

TABLE 1. CHARACTERISTICS OF A LAMINAR BOUNDARY LAYER FOR A LONGITU- /35
DINALLY STREAMLINED PLATE WITH UNIFORM SUCTION.

| $\sqrt{\xi}=\frac{-v_0}{U_0}\sqrt{\frac{U_0 x}{\nu}}$ | $\frac{-v_0\delta^*}{\nu}$ | $\frac{-v_0\delta^{**}}{\nu}$ | $H=\frac{\delta^*}{\delta^{**}}$ | $\frac{\tau_0\delta^*}{\mu U_0}$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 2,59 | 0,571 |
| 0,0707 | 0,114 | 0,045 | 2,53 | 0,607 |
| 0,141 | 0,211 | 0,086 | 2,47 | 0,631 |
| 0,212 | 0,303 | 0,125 | 2,43 | 0,671 |
| 0,283 | 0,381 | 0,160 | 2,39 | 0,699 |
| 0,354 | 0,450 | 0,192 | 2,35 | 0,726 |
| 0,424 | 0,511 | 0,221 | 2,31 | 0,750 |
| 0,495 | 0,566 | 0,248 | 2,28 | 0,773 |
| 0,566 | 0,614 | 0,273 | 2,25 | 0,794 |
| 0,636 | 0,658 | 0,295 | 2,23 | 0,813 |
| 0,707 | 0,695 | 0,315 | 2,21 | 0,830 |
| ∞ | 1 | 0,5 | 2,0 | 1 |

An approximate solution of this problem was found by Schlichting [100] and Thwaites [111]. The corresponding experimental investigations were carried out by Kay [81]. Figure 5 shows the results of these experimental investigations and a comparison of their theoretical computations. The comparison shows a good correlation between the results of the theoretical calculations and the experimental data.

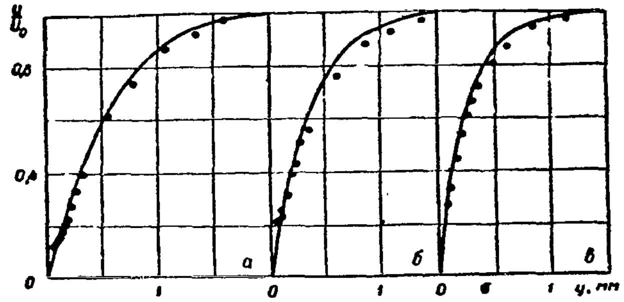



Fig. 5. Comparison of the Theoretical and Experimental Profiles of Velocity in a Laminar Boundary Layer in the Presence of Suction: a - $\xi$ = 0.77, $\frac{v_0}{U_0}$ = 0.00016; b - $\xi$ = 0.608, $\frac{v_0}{U_0}$ = 0.00045; c - $\xi$ = 1.87, $\frac{v_0}{U_0}$ = 0.00078.

The particular case of longitudinal streamlining of a flat plate with suction distributed according to the law /36

$$v_0(x) = -\frac{C}{2}\sqrt{\frac{\nu U_0}{x}}, \qquad (2.6)$$

where $C$ = const. was investigated by Schlichting and K. Bussman [102] by numerical integration of system (2.1). The curves, plotted as a result of their calculations of the velocity profiles, are shown in Figure 6 ($\gamma = \frac{2}{\sqrt{2}} v_0 \sqrt{\frac{x}{U_0 \nu}}$), and the numberical values of the fundamental local characteristics of the boundary layer are shown in Table 2.

TABLE 2. CHARACTERISTICS OF A LAMINAR BOUNDARY LAYER FOR A LONGITUDINALLY STREAMLINED PLATE WITH SUCTION ACCORDING TO THE LAW $v_0 \sim x^{-1/2}$

| $\frac{v_0}{U_0}\sqrt{\frac{U_0 x}{\nu}}$ | $\delta^*\sqrt{\frac{U_0}{\nu x}}$ | $\delta^{**}\sqrt{\frac{U_0}{\nu x}}$ | $H=\frac{\delta^*}{\delta^{**}}$ | $\frac{\tau_0 \delta^*}{\mu U_0}$ |
|---|---|---|---|---|
| 0 | 1,721 | 0,664 | 2,59 | 0,573 |
| 0,5 | 1,303 | 0,541 | 2,41 | 0,682 |
| 1 | 1,047 | 0,458 | 2,29 | 0,763 |
| 1,5 | 0,863 | 0,390 | 2,22 | 0,818 |

In the above-mentioned reference [102], Schlichting and Bussman

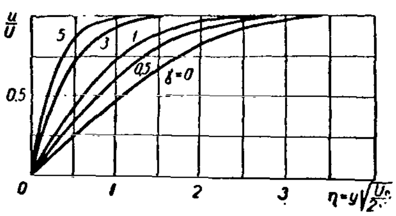



Fig. 6. Velocity Profiles on a Flat, Longitudinally Streamlined Plate with Suction Distributed According to the Law $x^{-1/2}$.

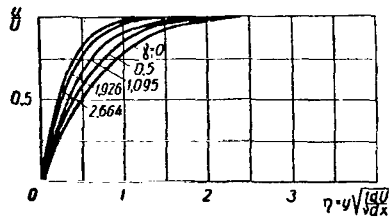



Fig. 7. Velocity Profiles on a Flat, Laterally Streamlined Plate.

also examined the flow near the leading critical point with lateral streamlining of a flat plate and uniform suction. In this case the longitudinal and lateral components of velocity in the potential flow are defined by the system /37

$$U(x) = u_1 x; \quad V(y) = -u_1 y.$$

The results of these investigations are shown in Figure 7, $\left(\gamma_1 = \sqrt{\frac{v_0}{\nu \frac{dU}{dx}}}\right)$ and in Table 3, respectively.

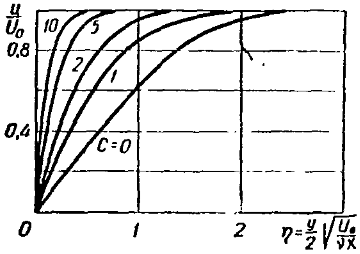



Fig. 8. Velocity Profiles in a Boundary Layer of a Plate (According to Emmons and Leigh).

Recently, H. Emmons and D. Leigh [62] published detailed tables of computer calculations of a system of equations for a boundary layer (2.1) for a longitudinally streamlined plate in the presence of suction distributed according to the law (2.6). Figure 8 and Table 4 show the data of the computations of velocity profiles for the case of suction. These data have practical value.

TABLE 3. CHARACTERISTICS OF A LAMINAR BOUNDARY LAYER FOR LATERALLY STREAMLINED PLATE WITH UNIFORM SUCTION.

| $\frac{-v_0}{\sqrt{\nu u_1}}$ | $\delta^*\sqrt{\frac{u_1}{\nu}}$ | $\delta^{**}\sqrt{\frac{u_1}{\nu}}$ | $H=\frac{\delta^*}{\delta^{**}}$ | $\frac{\tau_w}{\mu U_0}$ |
|---|---|---|---|---|
| 0 | 0,648 | 0,292 | 2,22 | 0,796 |
| 0,5 | 0,542 | 0,250 | 2,17 | 0,836 |
| 1,095 | 0,444 | 0,209 | 2,13 | 0,868 |
| 1,9265 | 0,349 | 0,167 | 2,09 | 0,917 |

## Power Distribution of Velocities Along the Outer Boundary of the Layer

We will consider the velocity profiles in the boundary layer to be self-similar if they can be expressed by a single function with an affine transformation. For a plane flow this allows us to reduce the system of equations in partial derivatives for a laminar boundary layer in the presence of a longitudinal velocity gradient on the outer boundary, to one ordinary differential equation. In this case the problem is simplified significantly. We will now examine a system composed of the equation for a laminar boundary layer (1.9) and the equation of continuity, which we can write in the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2};$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad (2.7)$$

TABLE 4. DATA FROM THE RESULTS OF CALCULATIONS OF THE FUNCTION $f'$ ($\eta$) FOR THE BOUNDARY LAYER OF A PLATE IN THE PRESENCE OF DISTRIBUTED SUCTION.[1] /38 /39

| | $f'$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\eta$ | $f(0)=0,00$, $f''(0)=1,32823$ | $f(0)=0,05$, $f''(0)=1,40103$ | $f(0)=0,10$, $f''(0)=1,47469$ | $f(0)=0,15$, $f''(0)=1,54919$ | $f(0)=0,20$, $f''(0)=1,62448$ | $f(0)=0,25$, $f''(0)=1,70054$ | $f(0)=0,30$, $f''(0)=1,77734$ | $f(0)=0,35$, $f''(0)=1,85484$ | $f(0)=0,40$, $f''(0)=1,93301$ |
| 0,00000 | 0,00000 | 0,00000 | 0,00000 | 0,00000 | 0,00000 | 0,00000 | 0,00000 | 0,00000 | 0,00000 |
| 0,10000 | 0,13282 | 0,13974 | 0,14673 | 0,15375 | 0,16082 | 0,16793 | 0,17508 | 0,18226 | 0,18947 |
| 0,20000 | 0,26553 | 0,27868 | 0,29187 | 0,30508 | 0,31832 | 0,33156 | 0,34481 | 0,35807 | 0,37131 |
| 0,30000 | 0,38788 | 0,41652 | 0,43513 | 0,45368 | 0,47218 | 0,49060 | 0,50894 | 0,52719 | 0,54534 |
| 0,40000 | 0,52942 | 0,55280 | 0,57602 | 0,59905 | 0,62190 | 0,64454 | 0,66696 | 0,68917 | 0,71114 |
| 0,50000 | 0,65956 | 0,68689 | 0,71388 | 0,74052 | 0,76679 | 0,79270 | 0,81822 | 0,84336 | 0,86810 |
| 0,60000 | 0,78755 | 0,81800 | 0,84791 | 0,87726 | 0,90605 | 0,93428 | 0,69193 | 0,98902 | 1,01554 |
| 0,70000 | 0,91252 | 0,94525 | 0,97721 | 1,00838 | 1,03878 | 1,06841 | 1,09727 | 1,12538 | 1,15273 |
| 0,80000 | 1,03352 | 1,06767 | 1,10080 | 1,13293 | 1,16407 | 1,19423 | 1,22342 | 1,25168 | 1,27901 |
| 0,90000 | 1,14952 | 1,18425 | 1,21773 | 1,24998 | 1,28103 | 1,31091 | 1,33965 | 1,36728 | 1,39384 |
| 1,00000 | 1,25953 | 1,29404 | 1,32708 | 1,35868 | 1,38890 | 1,41778 | 1,44537 | 1,47173 | 1,49689 |
| 1,10000 | 1,36262 | 1,39617 | 1,42804 | 1,45832 | 1,48706 | 1,51434 | 1,54021 | 1,56476 | 1,68803 |
| 1,20000 | 1,45797 | 1,48989 | 1,51999 | 1,54837 | 1,57512 | 1,60031 | 1,62404 | 1,64638 | 1,66741 |
| 1,30000 | 1,54491 | 1,57467 | 1,60252 | 1,62856 | 1,65291 | 1,67568 | 1,69696 | 1,71685 | 1,73544 |
| 1,40000 | 1,62302 | 1,65020 | 1,67543 | 1,69883 | 1,72055 | 1,74068 | 1,75936 | 1,77668 | 1,79275 |
| 1,50000 | 1,69209 | 1,71642 | 1,73881 | 1,75941 | 1,77836 | 1,79580 | 1,81184 | 1,82661 | 1,84019 |
| 1,60000 | 1,75216 | 1,77350 | 1,79297 | 1,81072 | 1,82693 | 1,84171 | 1,85520 | 1,86751 | 1,87876 |
| 1,70000 | 1,80352 | 1,82186 | 1,83844 | 1,85343 | 1,86699 | 1,87927 | 1,89037 | 1,90043 | 1,90953 |
| 1,80000 | 1,84666 | 1,86210 | 1,87593 | 1,88833 | 1,89945 | 1,90942 | 1,91837 | 1,92641 | 1,99363 |
| 1,90000 | 1,88224 | 1,89497 | 1,90628 | 1,91632 | 1,92525 | 1,93319 | 1,94025 | 1,94654 | 1,95214 |
| 2,00000 | 1,91104 | 1,92133 | 1,93038 | 1,93835 | 1,94537 | 1,95155 | 1,95701 | 1,96183 | 1,96609 |
| 2,10000 | 1,93391 | 1,94207 | 1,94916 | 1,95535 | 1,96076 | 1,96548 | 1,96961 | 1,97323 | 1,97639 |
| 2,20000 | 1,95174 | 1,95806 | 1,96352 | 1,96823 | 1,97230 | 1,97583 | 1,97889 | 1,98155 | 1,98386 |
| 2,30000 | 1,96537 | 1,97017 | 1,97427 | 1,97779 | 1,98080 | 1,98338 | 1,98560 | 1,98751 | 1,98916 |
| 2,40000 | 1,97558 | 1,97915 | 1,98218 | 1,98474 | 1,96692 | 1,98878 | 1,99035 | 1,99170 | 1,99285 |
| 2,50000 | 1,98308 | 1,98569 | 1,98788 | 1,98971 | 1,99125 | 1,99256 | 1,99365 | 1,99458 | 1,99537 |
| 2,60000 | 1,98849 | 1,99035 | 1,99190 | 1,99318 | 1,99426 | 1,99515 | 1,99590 | 1,99653 | 1,99706 |
| 2,70000 | 1,99231 | 1,99361 | 1,99469 | 1,99557 | 1,99630 | 1,99690 | 1,99740 | 1,99782 | 1,99816 |
| 2,80000 | 1,99496 | 1,99585 | 1,99658 | 1,99717 | 1,99766 | 1,99806 | 1,99838 | 1,99865 | 1,99887 |
| 2,90000 | 1,99675 | 1,99735 | 1,99784 | 1,99823 | 1,99854 | 1,99880 | 1,99901 | 1,99918 | 1,99932 |
| 3,00000 | 1,99795 | 1,99834 | 1,99866 | 1,99891 | 1,99911 | 1,99928 | 1,99941 | 1,99951 | 1,99960 |
| 3,10000 | 1,99873 | 1,99898 | 1,99918 | 1,99934 | 1,99947 | 1,99957 | 1,99965 | 1,99972 | 1,99977 |
| 3,20000 | 1,99922 | 1,99939 | 1,99951 | 1,99961 | 1,99969 | 1,99975 | 1,99980 | 1,99984 | 1,99987 |
| 3,30000 | 1,99954 | 1,99964 | 1,99971 | 1,99977 | 1,99982 | 1,99986 | 1,99988 | 1,99991 | 1,99993 |
| 3,40000 | 1,99973 | 1,99979 | 1,99984 | 1,99987 | 1,99990 | 1,99992 | 1,99994 | 1,99995 | 1,99996 |
| 3,50000 | 1,99984 | 1,99988 | 1,99991 | 1,99993 | 1,99994 | 1,99996 | 1,99996 | 1,99997 | 1,99998 |
| 3,60000 | 1,99991 | 1,99993 | 1,99995 | 1,99996 | 1,99997 | 1,99998 | 1,99998 | 1,99998 | 1,99999 |
| 3,70000 | 1,99995 | 1,99996 | 1,99997 | 1,99998 | 1,99998 | 1,99999 | 1,99999 | 1,99999 | 1,99999 |
| 3,80000 | 1,99997 | 1,99998 | 1,99999 | 1,99999 | 1,99999 | 1,99999 | 1,99999 | 2,00000 | 2,00000 |
| 3,90000 | 1,99999 | 1,99999 | 1,99999 | 1,99999 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 |
| 4,00000 | 1,99999 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 |
| 4,10000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 |
| 4,20000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 |
| 4,30000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 |
| 4,40000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 |
| 4,50000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | — | — | — |
| 4,60000 | 2,00000 | — | — | 2,00000 | — | — | — | — | — |
| 4,70000 | 2,00000 | — | — | 2,00000 | — | — | — | — | — |
| 4,80000 | 2,00000 | — | — | 2,00000 | — | — | — | — | — |
| 4,90000 | 2,00000 | — | — | 2,00000 | — | — | — | — | — |
| 5,00000 | 2,00000 | — | — | 2,00000 | — | — | — | — | — |

[1] When using Table 4, we should bear in mind that $f'=2\frac{u}{U_0}$, $\eta=\frac{1}{2}\left(\frac{U_0}{\nu}\right)\frac{y}{x^{1/2}}$.

TABLE 4 (Cont'd) /40 /41

| η | f′: f(0)=0,40, f″(0)=2,01184 | f(0)=0,50, f″(0)=2,09129 | f(0)=0,60, f″(0)=2,25198 | f(0)=0,70, f″(0)=2,41490 | f(0)=0,80, f″(0)=2,57988 | f(0)=0,90, f″(0)=2,74678 | f(0)=1,00, f″(0)=2,9154[illegible] | f(0)=1,10, f″(0)=3,0858[illegible] | f(0)=1,20, f″(0)=3,25772 |
|---|---|---|---|---|---|---|---|---|---|
| 0,00000 | 0,00000 | 0,00000 | 0,00000 | 0, 0000 | 0,00000 | 0,00000 | 0,00000 | 0,00000 | 0,00000 |
| 0,10000 | 0,19671 | 0,20397 | 0,21855 | 0,23321 | 0,24791 | 0,26265 | 0,27741 | 0,29218 | 0,30694 |
| 0,20000 | 0,38455 | 0,39776 | 0,42412 | 0,45036 | 0,47643 | 0,50233 | 0,52802 | 0,55348 | 0,57869 |
| 0,30000 | 0,56339 | 0,58133 | 0,61685 | 0,65186 | 0,68633 | 0,72022 | 0,75351 | 0,78619 | 0,81823 |
| 0,40000 | 0,73288 | 0,75437 | 0,79661 | 0,83783 | 0,87799 | 0,91709 | 0,95511 | 0,99205 | 1,02791 |
| 0,50000 | 0,89245 | 0,91640 | 0,96310 | 1,00818 | 1,05165 | 1,09353 | 1,13382 | 1,17256 | 1,20978 |
| 0,60000 | 1,04149 | 1,06687 | 1,11595 | 1,16281 | 1,20750 | 1,25008 | 1,29060 | 1,32914 | 1,36576 |
| 0,70000 | 1,17934 | 1,20621 | 1,25482 | 1,30164 | 1,34578 | 1,38736 | 1,43649 | 1,46328 | 1,49785 |
| 0,80000 | 1,30544 | 1,33098 | 1,37951 | 1,42477 | 1,46693 | 1,50618 | 1,54268 | 1,57660 | 1,60811 |
| 0,90000 | 1,41936 | 1,44388 | 1,49001 | 1,53250 | 1,57160 | 1,60754 | 1,64057 | 1,67090 | 1,69873 |
| 1,00000 | 1,52091 | 1,54382 | 1,58654 | 1,62537 | 1,66065 | 1,69268 | 1,72175 | 1,74811 | 1,77200 |
| 1,10000 | 1,61009 | 1,63100 | 1,66958 | 1,70419 | 1,73522 | 1,76303 | 1,78794 | 1,81025 | 1,83022 |
| 1,20000 | 1,68721 | 1,70584 | 1,73986 | 1,76997 | 1,79660 | 1,82015 | 1,84098 | 1,85939 | 1,87566 |
| 1,30000 | 1,75281 | 1,76903 | 1,79835 | 1,82393 | 1,84625 | 1,86572 | 1,88271 | 1,89754 | 1,91047 |
| 1,40000 | 1,80765 | 1,82147 | 1,84616 | 1,86741 | 1,88568 | 1,90141 | 1,91495 | 1,92661 | 1,93665 |
| 1,50000 | 1,85270 | 1,86420 | 1,88455 | 1,90180 | 1,91643 | 1,92884 | 1,93938 | 1,94834 | 1,95595 |
| 1,60000 | 1,88903 | 1,89840 | 1,91480 | 1,92850 | 1,93994 | 1,94952 | 1,95755 | 1,96427 | 1,96991 |
| 1,70000 | 1,91779 | 1,92526 | 1,93819 | 1,94883 | 1,95759 | 1,96482 | 1,97079 | 1,97573 | 1,97981 |
| 1,80000 | 1,94012 | 1,94596 | 1,95593 | 1,96402 | 1,97058 | 1,97592 | 1,98026 | 1,98381 | 1,98670 |
| 1,90000 | 1,95714 | 1,96160 | 1,96913 | 1,97614 | 1,97995 | 1,98381 | 1,98690 | 1,98939 | 1,99140 |
| 2,00000 | 1,96986 | 1,97319 | 1,97875 | 1,98313 | 1,98658 | 1,98931 | 1,99147 | 1,99318 | 1,99454 |
| 2,10000 | 1,97917 | 1,98161 | 1,98564 | 1,98876 | 1,99118 | 1,99307 | 1,99454 | 1,99569 | 1,99659 |
| 2,20000 | 1,98587 | 1,98761 | 1,99047 | 1,99265 | 1,99431 | 1,99559 | 1,99657 | 1,99733 | 1,99792 |
| 2,30000 | 1,99058 | 1,99181 | 1,99379 | 1,99528 | 1,99639 | 1,99724 | 1,99789 | 1,99838 | 1,99875 |
| 2,40000 | 1,99384 | 1,99468 | 1,99603 | 1,99702 | 1,99776 | 1,99831 | 1,99872 | 1,99903 | 1,99926 |
| 2,50000 | 1,99604 | 1,99661 | 1,99750 | 1,99816 | 1,99863 | 1,99898 | 1,99924 | 1,99943 | 1,99957 |
| 2,60000 | 1,99750 | 1,99788 | 1,99846 | 1,99888 | 1,99918 | 1,99940 | 1,99956 | 1,99967 | 1,99976 |
| 2,70000 | 1,99845 | 1,99869 | 1,99907 | 1,99933 | 1,99952 | 1,99965 | 1,99975 | 1,99982 | 1,99986 |
| 2,80000 | 1,99906 | 1,99921 | 1,99945 | 1,99961 | 1,99972 | 1,99980 | 1,99986 | 1,99990 | 1,99993 |
| 2,90000 | 1,99944 | 1,99953 | 1,99968 | 1,99978 | 1,99984 | 1,99989 | 1,99992 | 1,99994 | 1,99996 |
| 3,00000 | 1,99967 | 1,99973 | 1,99981 | 1,99987 | 1,99991 | 1,99994 | 1,99996 | 1,99997 | 1,99998 |
| 3,10000 | 1,99981 | 1,99984 | 1,99990 | 1,99993 | 1,99995 | 1,99997 | 1,99998 | 1,99998 | 1,99999 |
| 3,20000 | 1,99989 | 1,99991 | 1,99994 | 1,99996 | 1,99997 | 1,99998 | 1,99999 | 1,99999 | 1,99999 |
| 3,30000 | 1,99994 | 1,99995 | 1,99997 | 1,99998 | 1,99999 | 1,99999 | 1,99999 | 2,00000 | 2,00000 |
| 3,40000 | 1,99997 | 1,99997 | 1,99998 | 1,99999 | 1,99999 | 1,99999 | 2,00000 | 2,00000 | 2,00000 |
| 3,50000 | 1,99998 | 1,99999 | 1,99999 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 |
| 3,60000 | 1,99999 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 |
| 3,70000 | 1,99999 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 |
| 3,80000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 |
| 3,90000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | |
| 4,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | — | — | — |
| 4,10000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | 2,00000 | — | — | — | — |
| 4,20000 | 2,00000 | 2,00000 | 2,00000 | — | — | — | — | — | — |
| 4,30000 | 2,00000 | 2,00000 | — | — | — | — | — | — | — |

| η | f′: f(0)=1,30, f″(0)=3,43104 | f(0)=1,40, f″(0)=3,60574 | f(0)=1,50 f″(0)=3,78169 | f(0)=2,00, f″(0)=4,67770 |
|---|---|---|---|---|
| 0,00000 | 0,00000 | 0,00000 | 0,00000 | 0,00000 |
| 0,10000 | 0,32169 | 0,33642 | 0,35112 | 0,42387 |
| 0,20000 | 0,60364 | 0,62832 | 0,55271 | 0,77007 |
| 0,30000 | 0,84963 | 0,88039 | 0,91049 | 1,05116 |
| 0,40000 | 1,06270 | 1,09643 | 1,12911 | 1,27732 |
| 0,50000 | 1,24550 | 1,27977 | 1,31263 | 1,45721 |
| 0,60000 | 1,40053 | 1,43354 | 1,46485 | 1,59834 |
| 0,70000 | 1,53030 | 1,56077 | 1,58935 | 1,70739 |
| 0,80000 | 1,63735 | 1,66448 | 1,68964 | 1,79029 |
| 0,90000 | 1,72426 | 1,74767 | 1,76913 | 1,85220 |

TABLE 4 (Cont'd)

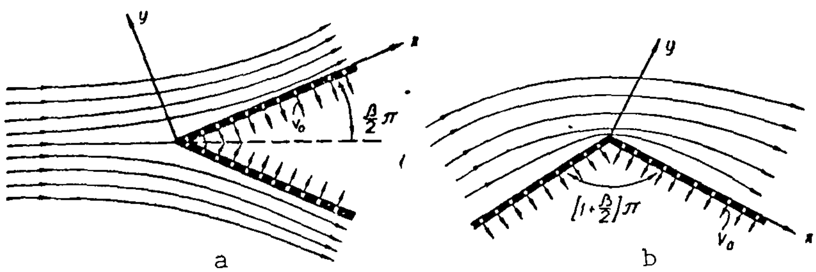
| $\eta$ | $f'$ | | | |
|---|---|---|---|---|
| | $f(0)=1{,}30$, $f''(0)=3{,}43104$ | $f(0)=1{,}40$, $f''(0)=3{,}60574$ | $f(0)=1{,}50$, $f''(0)=3{,}78169$ | $f(0)=2{,}00$, $f''(0)=4{,}67770$ |
| 1,00000 | 1,79365 | 1,81328 | 1,83305 | 1,89762 |
| 1,10000 | 1,84809 | 1,86410 | 1,87842 | 1,93032 |
| 1,20000 | 1,89004 | 1,90276 | 1,91400 | 1,95341 |
| 1,30000 | 1,92176 | 1,93162 | 1,94022 | 1,96941 |
| 1,40000 | 1,94529 | 1,95275 | 1,95918 | 1,98027 |
| 1,50000 | 1,96641 | 1,96793 | 1,97262 | 1,98751 |
| 1,60000 | 1,97464 | 1,97862 | 1,98197 | 1,92224 |
| 1,70000 | 1,98319 | 1,98601 | 1,98834 | 1,99527 |
| 1,80000 | 1,98906 | 1,99101 | 1,99260 | 1,99717 |
| 1,90000 | 1,99301 | 1,99433 | 1,99538 | 1,99834 |
| 2,00000 | 1,99561 | 1,99649 | 1,99718 | 1,99904 |
| 2,10000 | 1,99730 | 1,99786 | 1,99830 | 1,99946 |
| 2,20000 | 1,99836 | 1,99873 | 1,99900 | 1,99970 |
| 2,30000 | 1,99903 | 1,99925 | 1,99942 | 1,99984 |
| 2,40000 | 1,99943 | 1,99957 | 1,99967 | 1,99991 |
| 2,50000 | 1,99967 | 1,99976 | 1,99982 | 1,99995 |
| 2,60000 | 1,99981 | 1,99987 | 1,99990 | 1,99998 |
| 2,70000 | 1,99989 | 1,99993 | 1,99995 | 1,99999 |
| 2,80000 | 1,99994 | 1,99996 | 1,99997 | 1,99999 |
| 2,90000 | 1,99996 | 1,99998 | 1,99999 | 2,00000 |
| 3,00000 | 1,99998 | 1,99999 | 1,99999 | 2,00000 |
| 3,10000 | 1,99998 | 1,99999 | 2,00000 | 2,00000 |
| 3,20000 | 1,99999 | ,00000 | 2,00000 | 2,00000 |
| 3,30000 | 1,99999 | 2,00000 | 2,00000 | 2,00000 |
| 3,40000 | 1,99999 | 2,00000 | 2,00000 | 2,00000 |
| 3,50000 | 1,99999 | 2,00000 | 2,00000 | — |
| 3,60000 | 1,99999 | 2,00000 | 2,00000 | — |
| 3,70000 | 1,99999 | 2,00000 | 2,00000 | — |
| 3,80000 | 1,99999 | — | — | — |
| 3,90000 | 1,99999 | — | — | — |
| 4,00000 | 1,99999 | — | — | — |

| $\eta$ | $f'$ | | | | | |
|---|---|---|---|---|---|---|
| | $f(0)=2{,}50$, $f''(0)=05{,}5953$ | $f(0)=3{,}00$, $f''(0)=06{,}5289$ | $f(0)=4{,}00$, $f''(0)=08{,}4296$ | $f(0)=5{,}00$, $f''(0)=10{,}3590$ | $f(0)=6{,}00$, $f''(0)=12{,}3083$ | $f(0)=10{,}00$, $f''(0)=20{,}1941$ |
| 0,00000 | 0,00000 | 0,00000 | 0,00000 | 0,00000 | 0,00000 | 0,00000 |
| 0,10000 | 0,49497 | 0,56392 | 0,69455 | 0,81494 | 0,92514 | 1,27525 |
| 0,20000 | 0,87935 | 0,98034 | 1,15830 | 1,30696 | 1,43026 | 1,74147 |
| 0,30000 | 1,17585 | 1,28551 | 1,46513 | 1,60095 | 1,70290 | 1,90944 |
| 0,40000 | 1,40223 | 1,50665 | 1,66551 | 1,77411 | 1,84780 | 1,96888 |
| 0,50000 | 1,57286 | 1,66469 | 1,79437 | 1,87440 | 1,92345 | 1,98951 |
| 0,60000 | 1,69960 | 1,77586 | 1,87580 | 1,93144 | 1,96222 | 1,99653 |
| 0,70000 | 1,79220 | 1,85274 | 1,92634 | 1,96327 | 1,98171 | 1,99888 |
| 0,80000 | 1,85870 | 1,90495 | 1,95712 | 1,98069 | 1,99131 | 1,99964 |
| 0,90000 | 1,90558 | 1,93974 | 1,97550 | 1,99004 | 1,99595 | 1,99989 |

## TABLE 4 (Cont'd) /44

| $\eta$ | $f'$ | | | | | |
|---|---|---|---|---|---|---|
| | $f(0)=2{,}50$, $f''(0)=$ $=05{,}5953$ | $f(0)=3{,}00$, $f''(0)=$ $=06{,}5289$ | $f(0)=4{,}00$, $f''(0)=$ $=08\ 4296$ | $f(0)=5{,}00$, $f''(0)=$ $=10{,}3596$ | $f(0)=6{,}00$, $f''(0)=$ $=12{,}3083$ | $f(0)=10{,}00$, $f''(0)=$ $=20{,}1941$ |
| 1,00000 | 1,93803 | 1,96250 | 1,98626 | 1,99496 | 1,99815 | 1,99997 |
| 1,10000 | 1,96006 | 1,97708 | 1,99244 | 1,99750 | 1,99917 | 1,99999 |
| 1,20000 | 1,97472 | 1,98626 | 1,99592 | 1,99878 | 1,99964 | 2,00000 |
| 1,30000 | 1,98429 | 1,99191 | 1,99784 | 1,99942 | 1,99984 | 2,00000 |
| 1,40000 | 1,99042 | 1,99533 | 1,99888 | 1,99973 | 1,99993 | 2,00000 |
| 1,50000 | 1,99427 | 1,99735 | 1,99943 | 1,99987 | 1,99997 | — |
| 1,60000 | 1,99663 | 1,99853 | 1,99971 | 1,99994 | 1,99999 | — |
| 1,70000 | 1,99806 | 1,99920 | 1,99986 | 1,99998 | 2,00000 | — |
| 1,80000 | 1,99890 | 1,99957 | 1,99993 | 1,99999 | 2,00000 | — |
| 1,90000 | 1,99939 | 1,99977 | 1,99997 | 2,00000 | 2,00000 | — |
| 2,00000 | 1,99967 | 1,99988 | 1,99999 | 2,00000 | — | — |
| 2,10000 | 1,99982 | 1,99994 | 1,99999 | — | — | — |
| 2,200000 | 1,99991 | 1,99997 | 2,00000 | — | — | — |
| 2,30000 | 1,99995 | 1,99999 | 2,00000 | — | — | — |
| 2,40000 | 1,99998 | 1,99999 | 2,00000 | — | — | — |
| 2,50000 | 1,99999 | 2,00000 | — | — | — | — |
| 2,60000 | 1,99999 | 2,00000 | — | — | — | — |
| 2,70000 | 2,00000 | 2,00000 | — | — | — | — |
| 2,80000 | 2,00000 | — | | — | — | — |
| 2,90000 | 2,00000 | — | | — | — | — |

/45




a b

Fig. 9. A Geometric Interpretation of the Self-Similar Solution a is for $m > 0$, the Accelerated Flow, b is for $m < 0$, the Decelerated Flow.

We will now investigate the case in which the velocity on the outer boundary of the boundary layer is determined by the power inequality[1]

$$U(x) = Cx^m, \tag{2.8}$$

[1] The self-similar problem of a laminar boundary layer on a permeable surface is examined also in reference [57].

where $C$ = const.; with $m > 0$ (Fig. 9a) on the outer boundary of the layer the current is accelerated, and with $m < 0$ (Fig. 9b) the flow is decelerated.

Let us now introduce the current function $\psi(x,y)$. Then the equation of continuity of system (2.7) is automatically satisfied, since

$$u=\frac{\partial\psi}{\partial y};\quad v=-\frac{\partial\psi}{\partial x}. \tag{2.9}$$

Using the new variables

$$\psi=\sqrt{\frac{2}{m+1}}\sqrt{\nu Cx^{m+1}}f_0(\eta); \tag{2.10}$$

$$\eta=y\sqrt{\frac{m+1}{2}}\sqrt{\frac{Cx^{m-1}}{\nu}};$$

after the necessary computations we determine that /46

$$\left.\begin{aligned}
&u=Cx^m f_0'(\eta);\\
&\frac{\partial u}{\partial y}=Cx^m f_0''(\eta)\sqrt{\frac{m+1}{2}}\sqrt{\frac{Cx^{m-1}}{\nu}};\\
&\frac{\partial^2 u}{\partial y^2}=Cx^m f_0'''(\eta)\frac{m+1}{2}\frac{Cx^{m-1}}{\nu};\\
&\frac{\partial u}{\partial x}=Cx^{m-1}\left(mf_0'+\frac{m-1}{2}\eta f_0''\right);\\
&v=-\sqrt{\frac{2}{m+1}}\sqrt{\nu Cx^{m+1}}\left(\frac{m+1}{2}f_0+\frac{m-1}{2}\eta f_0'\right).
\end{aligned}\right| \tag{2.11}$$

The symbol $'$ denotes derivatives with respect to $\eta$.

Substituting expression (2.11) into the first equation of system (2.7), we find that

$$f_0'''+f_0f_0''+\beta[1-(f_0')^2]=0, \tag{2.12}$$

where

$$\beta=\frac{2m}{m+1}\quad\text{or}\quad m=\frac{\beta}{2-\beta}.$$

This equation was first suggested by Falkner and Skan [63].

We will solve equation (2.12) with satisfaction of the following boundary conditions:

$$\begin{aligned} &\text{with } \eta = 0 \qquad f_0 = C_0;\; f_0' = 0; \\ &\text{with } \eta \to \infty \quad f_0' \to 1. \end{aligned} \tag{2.13}$$

Substituting the boundary conditions (2.13) into the last expression of (2.11), we obtain

$$v_0(x) = -C_0 \sqrt{\frac{\nu C}{2-\beta}}\, x^{\frac{-1-\beta}{2-\beta}}. \tag{2.14}$$

Computing the values of $f_0$, $f_0'$, $f_0''$ and $f_0'''$, we can determine all the characteristics of a laminar boundary layer from the following formulas: /47

$$\left.\begin{aligned} &\delta^{*} = \alpha \sqrt{\frac{2}{m+1}} \sqrt{\frac{\nu x}{U}}; \\ &\delta^{**} = \gamma \sqrt{\frac{2}{m+1}} \sqrt{\frac{\nu x}{U}} \\ &\tau_0 = f_0''(0)\mu U \sqrt{\frac{m+1}{2}} \sqrt{\frac{U}{\nu x}}; \\ &\frac{u}{U} = f_0'(\eta); \\ &\frac{y}{\delta^{**}} = \frac{\eta}{\gamma}; \\ &\frac{\delta^{**}}{U}\frac{\partial u}{\partial y} = \gamma f_0''(\eta); \\ &\frac{v\delta^{**}}{\nu} = -\gamma \frac{2}{m+1}\left[\frac{m+1}{2} f_0(\eta) + \frac{m-1}{2}\eta f_0'(\eta)\right]. \end{aligned}\right\} \tag{2.15}$$

H. Schaeffer integrated equation (2.12) by numerical method with the boundary conditions (2.13) for the particular case of β = 0.2. Complete tables of the functions $f_0$ and also their first, second and third derivatives can be found in reference [98]. Figure 10 shows the curves of the velocity profiles

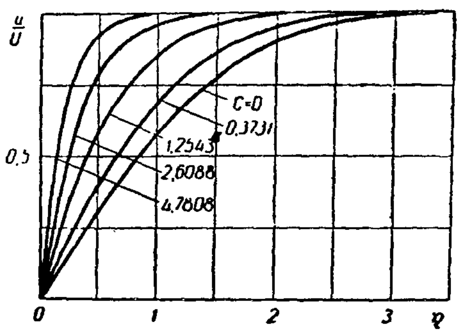



Fig. 10. Velocity Profiles for Self-Similar Solutions of the Equation for a Laminar Boundary Layer in the Presence of Suction.

for different values of the parameter $C$ according to the data of these calculations.

Reference [19] proposes a method for numerically integrating the equations of (2.12) on an M-20 computer. We can reduce the problem of (2.12) and (2.13) to a Cauchy problem. To do this, we will convert equation (2.12) into the form

$$f_0''' + [f_0(f_0' - 1)]' = (\beta + 1) f_0' (f_0' - 1) + \beta (f_0' - 1). \qquad (2.16)$$ /48

Integrating equation (2.16) over $\eta$ and considering that

$$f_0(\eta) [f_0'(\eta) - 1] \to 0 \quad \text{при} \quad \eta \to \infty,$$

we obtain

$$f_0''(0) = C - (\beta + 1) \int_0^\infty f_0' (f_0 - 1) \, d\eta - \beta \int_0^\infty (f_0' - 1) \, d\eta. \qquad (2.17)$$

We will take as a first approximation to the solution of problems (2.12) and (2.13) the function

$$\varphi(\eta) = b + \eta + ae^{-h\eta},$$

where the constants $h$, $b$ and $a$ are chosen from equation (2.12) and the boundary conditions (2.13). We will compute three first derivatives with respect to $\eta$ of the function $\phi(\eta)$:

$$\varphi' = 1 - ahe^{-h\eta}; \quad \varphi'' = ah^2 e^{-h\eta}; \quad \varphi''' = -ah^3 e^{-h\eta}$$

and, assuming that $\eta = 0$, we find that

$$\varphi(0) = b + a; \quad \varphi'(0) = 1 - ah; \quad \varphi''(0) = ah^2; \quad \varphi'''(0) = -ah^3.$$

Satisfying the condition (2.13), we obtain

$$a + b = c; \qquad ah = 1;$$
$$\varphi''(0) = h; \qquad \varphi'''(0) = -h^2.$$

Substituting these relationships into the differential equation (2.12), we obtain the quadratic algebraic equation for determining $h$:

$$h^2 - ch - \beta = 0.$$

From this equation, we get

$$h_1 = \frac{C}{2} + \sqrt{\frac{C^2}{4} + \beta}. \tag{2.18}$$

When the expression under the root sign is less than zero, we will take

$$h_1 = \frac{C}{2}.$$

Substituting the value of the function $\phi'(\eta)$ into equation (2.16), we have the following expression for determining $h$: /49

$$h = C + \frac{(\beta + 1)}{h} + \frac{\beta}{h}$$

or

$$h_2 = \frac{C}{2} + \sqrt{\frac{C^2}{4} + \beta + \frac{(\beta + 1)}{2}}. \tag{2.19}$$

When the expression under the root sign is negative, then $h_2 = \frac{C}{2}$.

As a result, we take as a first approximation of $f_0''(0)$ the expression

$$f_0''(0) = \frac{h_1 + h_2}{2}. \tag{2.20}$$

Introducing the designations $f_0(\eta) = y_1(\eta)$, $\dot{y}_1(\eta) = y_2(\eta)$, $\dot{y}_2(\eta) = y_3(\eta)$, where the dot represents derivatives with respect to $\eta$, we can rewrite equation (2.17) as the system

$$\left.\begin{aligned} &\dot{y}_1(\eta) = y_2(\eta); \\ &\dot{y}_2(\eta) = y_3(\eta); \\ &\dot{y}_3(\eta) = -y_1(\eta)\, y_3(\eta) + \beta\,[y_2^2(\eta) - 1]. \end{aligned}\right\} \tag{2.21}$$

The boundary conditions (2.13) then take the form

$$\text{with } \eta \to \infty \quad y_1(0) = C; \quad y_2(0) = 0; \quad y_2(\eta) \to 1.$$

We will denote by $\bar{y}_1(\eta)$; $\bar{y}_2(\eta)$; $\bar{y}_3(\eta)$ the solution of system (2.21) with the initial conditions

$$\bar{y}_1(0) = C; \quad \bar{y}_2(0) = 0; \quad \bar{y}_3(0) = \frac{h_1 - h_2}{2}.$$

Furthermore, we will posit that $y_i(\eta) = \bar{y}_i(\eta) + \Delta y_i(\eta)$, where $i$ = 1, 2, 3. Let us substitute these equations into the system (2.21) and, considering only first order differences, we obtain the system [38]

$$\left.\begin{aligned} &\frac{d\Delta y_1}{d\eta} = \Delta y_2; \\ &\frac{d\Delta y_2}{d\eta} = \Delta y_3; \\ &\frac{d\Delta y_3}{d\eta} = -\bar{y}_1\Delta y_3 - \bar{y}_3\Delta y_1 + 2\beta\bar{y}_2\Delta y_2 \end{aligned}\right\} \qquad (2.22)$$

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or in matrix form

$$\dot{\Delta} y = A(\eta)\Delta Y, \qquad (2.23)$$

where

$$A(\eta) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\bar{y}_3 & 2\beta\,\bar{y}_2 & -\bar{y}_1 \end{pmatrix}; \qquad \Delta Y = \begin{pmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{pmatrix},$$

and the dot represents derivatives with respect to $\eta$ from the column $\Delta Y$.

In addition to equation (2.23) we will examine the conjugate system

$$-\dot{X}(\eta) = \bar{A}(\eta)X(\eta), \qquad (2.24)$$

where $X(\eta) = \begin{matrix} X_1 \\ X_2 \\ X_3 \end{matrix}$, and $\bar{A}(\eta)$ is the transposition of the matrix. We will solve the system (2.24) with the following boundary conditions:

$$x_3(0) = 1; \qquad x_2(\eta_0) = x_3(\eta_0) = 0,$$

where we determine the value of $\eta_0$ from the condition of prescribing the accuracy, i.e., if $\varepsilon > 0$ (the prescribed accuracy), then we choose $\eta_0$ such that for all $\eta \geq \eta_0$

$$|\bar{y}_2(\eta)| < \varepsilon.$$

Since ($X$ is a matrix row)

$$(\Delta Y X)^{\cdot} = \dot{\Delta} X + \Delta Y \dot{\bar{X}} = A(\eta)\,\Delta Y X - \Delta Y \bar{A}(\eta)\, X = 0$$

or

$$\Delta \dot{y}_1 x_1 + \Delta y_2 x_2 + \Delta y_3 x_3 \Big|_0^{\eta_0} = 0,$$

then, since /51

$$\Delta y_1^{\cdot}(0) = \Delta y_2(0) = 0; \qquad x_3(0) = 1; \qquad x_1(\eta_0) = x_3(\eta_0) = 0,$$

we have

$$\Delta y_3(0) = x_2(\eta_0)\,\Delta y_2(\eta_0),$$

where

$$\Delta y_2(\eta_0) = 1 - \bar{y}_2(\eta_0).$$

In other words, to obtain each successive approximation, we found the increment $\Delta y_3(0)$ under the condition that $x_2(\eta_0)$ is known.

To determine $x_2(\eta_0)$ we will simultaneously solve the system of twelfth order equations (2.21) and (2.24), in which the first three equations are a reference system and the remaining nine allow us to reduce the boundary problem over $x(\eta)$ to a Cauchy problem [37]:

$$\left.\begin{aligned}
&\dot{\bar{y}}_1 = \bar{y}_2;\\
&\dot{y}_2 = y_3;\\
&\dot{\bar{y}}_3 = -\bar{y}_1\bar{y}_3 + \beta(\bar{y}_2^2 - 1);\\
&g_1 = \bar{y}_3 g_3;\\
&\dot{g}_2 = -g_1 - 2\beta\bar{y}_2 g_3;\\
&g_3 = -g_2 + \bar{y}_1 g_3;\\
&\dot{z}_1 = y_3 z_3;\\
&\dot{z}_2 = -z - 2\beta\bar{y}_2 z_3;\\
&\dot{z}_3 = -z_2 + \bar{y}_1 z_3;\\
&p_1 = \bar{y}_3 p_3;\\
&p_2 = -p_1 - 2\beta\bar{y}_2 p_3;\\
&\dot{p}_3 = -p_2 + \bar{y}_1 p_3
\end{aligned}\right| \qquad (2.25)$$

with the following initial conditions: /52

$$\bar{y}_1(0) = C; \qquad \bar{y}_2(0) = 0; \qquad \bar{y}_3(0) = \frac{h_1 + h_2}{2} + \Delta y_3'(0).$$

where $j$ is the number of the approximation;

$$g_1(0) = z_2(0) = p_3(0) = 1;$$
$$g_2(0) = g_3(0) = z_1(0) = z_3(0) = p_1(0) = p_2(0) = 0.$$

Then from the conditions

$$\left.\begin{aligned} x_1(\eta_0) &= c_1 g_1(\eta_0) + c_2 z_1(\eta_0) + c_3 p_1(\eta_0) = 0; \\ x_3(\eta_0) &= c_1 g_3(\eta_0) + c_2 z_2(\eta_0) + c_3 p_3(\eta_0) = 0; \\ x_3(0) &= c_1 g_3(0) + c_2 z_3(0) + c_3 p_3(0) = 0 \end{aligned}\right\} \tag{2.26}$$

we find that

$$x_2(\eta_0) = d_1 g_2(\eta_0) + d_2 z_2(\eta_0) + p_2(\eta_0),$$

where

$$d_1 = \frac{p_3(\eta_0) z_1(\eta_0) - p_1(\eta_0) z_3(\eta_0)}{g_1(\eta_0) z_3(\eta_0) - z(\eta_0) g_3(\eta_0)};$$
$$d_2 = \frac{p_1(\eta_0) g_3(\eta_0) - g_1(\eta_0) p_3(\eta_0)}{g_1(\eta_0) z_3(\eta_0) - z_1(\eta_0) g_3(\eta_0)}.$$

In the case of a permeable surface in the presence of suction of a fluid from the boundary layer the solution obtained proves to be unique (cf. reference [19]).

Using the proposed method, systematic calculations of the boundary layer characteristics were made for the case when the normal component of the velocity on the surface of the body is non-zero. Figure 11 shows the results of calculations of the integral and local characteristics of the boundary layer for different values of the parameter $C$:

$$\alpha(\beta, C) = \int_0^\infty [1 - f_0'(\eta)]\, d\eta; \quad \gamma(\beta, C) = \int_0^\infty [1 - f_0'(\eta)] f_0'(\eta)\, d\eta \quad \text{и} \quad f_0''(0).$$

The more general case, when not only the normal but also the tangential component of velocity on the surface of the body are non-zero, is examined in detail in reference [34].

### Exponential Velocity Distribution Along the Outer Boundary of the Layer /53

Let us now examine in more detail [23], the case of a boundary layer with an exponential law of velocity distribution in the outer

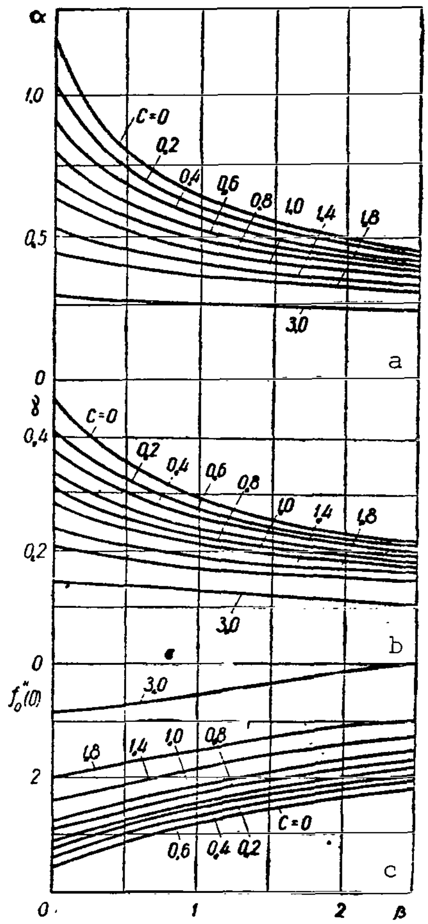



Fig. 11. The Dependence of the Parameters α (a), γ (b), and $f_0''(0)$ (c) on the Value of the Constant $C$.

flow, i.e., when

$$U = U_0 \exp lx, \tag{2.27}$$

where $U_0$ and $l$ are constants.

We now introduce in place of dimensional velocities and coordinates the dimensionless current function $\psi$ and the coordinate $\eta$:

$$\psi = \left(\frac{2\nu U}{l}\right)^{\frac{1}{2}} f_0(\eta); \tag{2.28}$$

$$\eta = \left(\frac{Ul}{2\nu}\right)^{\frac{1}{2}} y.$$

Then the longitudinal velocity in the boundary layer will be

$$u = Uf_0'. \tag{2.29}$$

Satisfying the equation of continuity (the second equation of system (2.1)), we obtain

$$v = -\left(\frac{\nu Ul}{2}\right)[f_0(\eta) - \eta f_0']. \tag{2.30}$$

Substituting expressions (2.28) - (2.30) into equation (2.7), we will have a third-order, nonlinear ordinary differential equation:

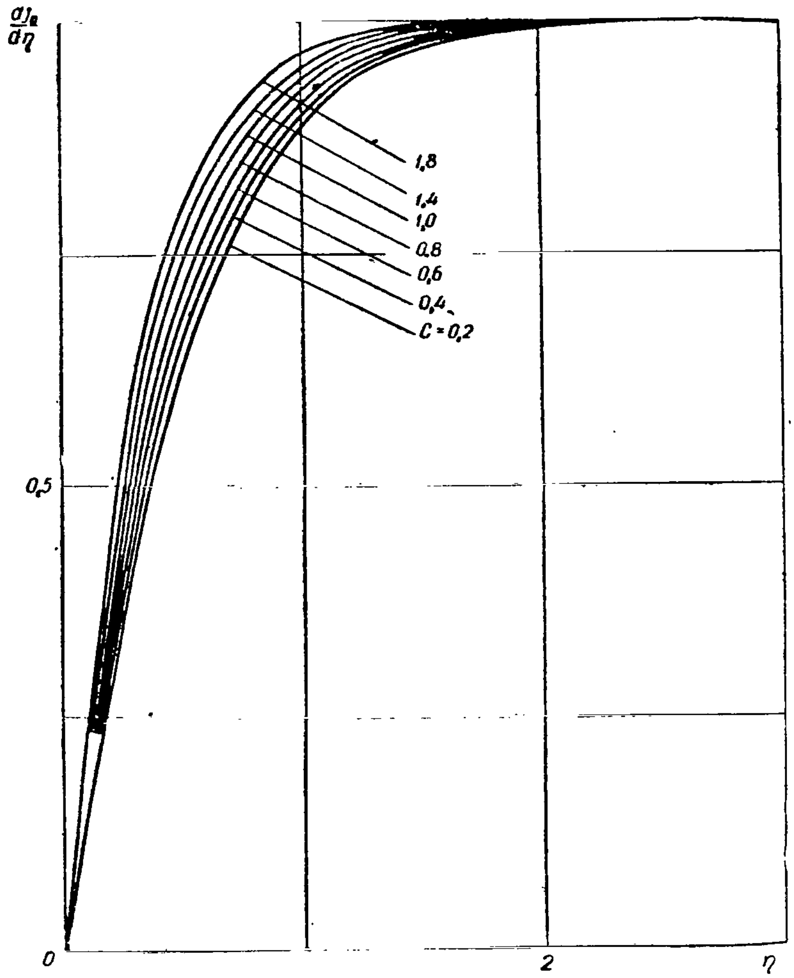
$$f_0''' + f_0 f_0'' + 2[1 - f_0'^2] = 0, \tag{2.31}$$

and the boundary conditions have the form

$$f_0 = -C; \quad f_0' = 0 \text{ with } \eta = 0; \quad f_0' \to 1 \text{ with } \eta \to \infty. \tag{2.32}$$

Here

$$C = \frac{v_0}{\frac{\nu Ul}{2}}. \tag{2.33}$$




/54

Fig. 12. Velocity Distribution Across the Boundary Layer with Exponential External Flow.

Equation (2.31) was numerically integrated on an M-20 computer by the method proposed above, with satisfaction of the boundary conditions for $C$ equal to 0.2; 0.4; 0.6; 0.8; 1.0; 1.4 and 1.8. Analogous data for a nonpermeable surface ($C$ = 0) were obtained in reference [72]. /55

As a result of these calculations, numerical values of the functions $f_0(\eta)$, $f_0'(\eta)$ and $f_0''(\eta)$ were determined and also of the following integral characteristics of the boundary layer:

$$A=\int_0^\infty (1-f_0')\,d\eta;\quad B=\int_0^\infty (1-f_0')\,f_0'd\eta;$$
$$D=\int_0^\infty (1-f_0'^2)\,f_0'd\eta;\quad F_0=\int_0^\infty f_0''^2 d\eta; \tag{2.34}$$

TABLE 5. VALUES OF THE FUNCTION $f_0$ (2.28) AND OF ITS FIRST AND SECOND DERIVATIVES. /56 /57

| η | C=0,2 | | | C=0,4 | | | C=0,6 | | | C=0,8 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $f_0$ | $f_0'$ | $f_0''$ | $f_0$ | $f_0'$ | $f_0''$ | $f_0$ | $f_0'$ | $f_0''$ | $f_0$ | $f_0'$ | $f_0''$ |
| 0,0 | 0,20000 | 0,00000 | 1,80162 | 0,40000 | 0,00000 | 1,9214 | 0,60000 | 0,00000 | 2,04611 | 0,80000 | 0,00000 | 2,17562 |
| 0,2 | 0,23294 | 0,31431 | 1,34965 | 0,43483 | 0,33097 | 1,40112 | 0,63677 | 0,34787 | 1,45094 | 0,83875 | 0,36492 | 1,49879 |
| 0,4 | 0,32010 | 0,54444 | 0,96518 | 0,52603 | 0,56678 | 0,97443 | 0,73201 | 0,58880 | 0,98039 | 0,93800 | 0,61039 | 0,88309 |
| 0,6 | 0,44614 | 0,70598 | 0,66411 | 0,55656 | 0,72771 | 0,65163 | 0,86686 | 0,74853 | 0,63667 | 1,07708 | 0,76834 | 0,61948 |
| 0,8 | 0,69902 | 0,81536 | 0,44182 | 0,81346 | 0,83360 | 0,42114 | 1,02759 | 0,85057 | 0,39940 | 1,24135 | 0,86626 | 0,37695 |
| 1,0 | 0,76978 | 0,88708 | 0,28504 | 0,98744 | 0,90104 | 0,26381 | 1,20453 | 0,91365 | 0,24275 | 1,42099 | 0,92496 | 0,22214 |
| 1,2 | 0,95212 | 0,93273 | 0,17862 | 1,17216 | 0,94272 | 0,16043 | 1,39137 | 0,95146 | 0,14316 | 1,60971 | 0,95007 | 0,12695 |
| 1,4 | 1,14172 | 0,96096 | 0,10881 | 1,36343 | 0,96772 | 0,09478 | 1,58407 | 0,97346 | 0,08196 | 1,80363 | 0,97830 | 0,07040 |
| 1,6 | 1,33576 | 0,97794 | 0,06444 | 1,55856 | 0,98230 | 0,05440 | 1,78012 | 0,98589 | 0,04557 | 2,00045 | 0,98882 | 0,03788 |
| 1,8 | 1,53243 | 0,98787 | 0,03710 | 1,75593 | 0,99056 | 0,03033 | 1,97806 | 0,99271 | 0,02459 | 2,19884 | 0,99441 | 0,01978 |
| 2,0 | 1,73062 | 0,99351 | 0,02075 | 1,95454 | 0,99511 | 0,01642 | 2,17700 | 0,99634 | 0,01288 | 2,39804 | 0,99728 | 0,01001 |
| 2,2 | 1,92967 | 0,99662 | 0,01127 | 2,15883 | 0,99754 | 0,00863 | 2,37647 | 0,99822 | 0,00974 | 2,59766 | 0,99872 | 0,00491 |
| 2,4 | 2,12918 | 0,99829 | 0,00594 | 2,35348 | 0,99880 | 0,00440 | 2,57623 | 0,99916 | 0,00322 | 2,79748 | 0,99942 | 0,00234 |
| 2,6 | 2,32893 | 0,99916 | 0,00304 | 2,55331 | 0,99843 | 0,00217 | 2,77611 | 0,99961 | 0,00154 | 2,99739 | 0,99974 | 0,00107 |
| 2,8 | 2,52881 | 0,99960 | 0,00151 | 2,75323 | 0,99974 | 0,00104 | 2,97606 | 0,99983 | 0,00071 | 3,19736 | 0,99989 | 0,00048 |
| 3,0 | 2,72876 | 0,99982 | 0,00072 | 2,95320 | 0,99988 | 0,00048 | 3,17603 | 0,99992 | 0,00032 | 3,39735 | 0,99995 | 0,00021 |
| 3,2 | 2,92873 | 0,99992 | 0,00034 | 3,15318 | 0,99995 | 0,00022 | 3,37602 | 0,99997 | 0,00014 | 3,59734 | 0,99998 | 0,00009 |
| 3,4 | 3,12872 | 0,99996 | 0,00015 | 3,35317 | 0,99998 | 0,00009 | 3,57602 | 0,99999 | 0,00006 | 3,79734 | 0,99999 | 0,00003 |
| 3,6 | 3,32872 | 0,99999 | 0,00007 | 3,55317 | 0,99999 | 0,00004 | 3,77602 | 0,99999 | 0,00002 | 3,99734 | 0,99999 | 0,00001 |
| 3,8 | 3,52871 | 0,99999 | 0,00002 | 3,75317 | 0,99999 | 0,00001 | 3,97602 | 0,99999 | 0,00001 | 4,19734 | 0,99999 | 0,00000 |
| 4,0 | 3,72871 | 0,99999 | 0,00001 | 3,95317 | 1,00000 | 0,00000 | 4,17602 | 0,99999 | 0,00000 | 4,39734 | 1,00000 | 0,00000 |
| 4,2 | 3,92871 | 1,00000 | 0,00000 | — | — | — | 4,37602 | 1,00000 | 0,00000 | — | — | — |

| η | C=1,0 | | | C=1,4 | | | C=1,8 | | |
|---|---|---|---|---|---|---|---|---|---|
| | $f_0$ | $f_0'$ | $f_0''$ | $f_0$ | $f_0'$ | $f_0''$ | $f_0$ | $f_0'$ | $f_0''$ |
| 0,0 | 1,00000 | 0,00000 | 2,30381 | 1,40000 | 0,00000 | 2,59061 | 1,80000 | 0,00000 | 2,88661 |
| 0,2 | 1,04076 | 0,38208 | 1,54438 | 1,44485 | 0,41641 | 1,62785 | 1,84900 | 0,45042 | 1,69991 |
| 0,4 | 1,14399 | 0,63147 | 0,98258 | 1,55585 | 0,67178 | 0,97238 | 1,96745 | 0,70928 | 0,95108 |
| 0,6 | 1,28712 | 0,78712 | 0,60036 | 1,70655 | 0,82141 | 0,55754 | 2,12498 | 0,85136 | 0,51064 |
| 0,8 | 1,45471 | 0,88067 | 0,35412 | 1,88009 | 0,90581 | 0,30846 | 2,30356 | 0,92640 | 0,26442 |
| 1,0 | 1,63679 | 0,93508 | 0,20220 | 2,06633 | 0,95180 | 0,16509 | 2,49311 | 0,96468 | 0,13240 |
| 1,2 | 1,82715 | 0,96563 | 0,11192 | 2,25938 | 0,97607 | 0,08558 | 2,63816 | 0,98357 | 0,06415 |
| 1,4 | 2,02212 | 0,98235 | 0,60 | 2,45597 | 0,98847 | 0,04298 | 2,88589 | 0,99259 | 0,03009 |
| 1,6 | 2,21957 | 0,99119 | 0,03128 | 2,65436 | 0,99461 | 0,02091 | 3,08488 | 0,99676 | 0,01366 |
| 1,8 | 2,41832 | 0,99574 | 0,01579 | 2,85361 | 0,99756 | 0,00985 | 3,28444 | 0,99863 | 0,00599 |
| 2,0 | 2,61772 | 0,99799 | 0,00772 | 3,05328 | 0,99893 | 0,00449 | 3,48427 | 0,99944 | 0,00255 |
| 2,2 | 2,81744 | 0,99909 | 0,00366 | 3,25314 | 0,99954 | 0,00198 | 3,68419 | 0,99978 | 0,00105 |
| 2,4 | 3,01732 | 0,99960 | 0,00168 | 3,45307 | 0,99981 | 0,00084 | 3,88416 | 0,99991 | 0,00042 |
| 2,6 | 3,21726 | 0,99983 | 0,00074 | 3,65305 | 0,99925 | 0,00035 | 4,08415 | 0,89997 | 0,00016 |
| 2,8 | 3,41724 | 0,99993 | 0,00032 | 3,85304 | 0,99997 | 0,00014 | 4,28415 | 0,99999 | 0,00006 |
| 3,0 | 3,61723 | 0,99997 | 0,00013 | 4,05303 | 0,99999 | 0,00005 | 4,48415 | 0,99999 | 0,00002 |
| 3,2 | 3,81722 | 0,99999 | 0,00005 | 4,25304 | 0,99999 | 0,00002 | 4,68415 | 0,99999 | 0,00001 |
| 3,4 | 4,01722 | 0,99999 | 0,00002 | 4,45304 | 0,99999 | 0,00000 | 4,88415 | 0,99999 | 0,00000 |
| 3,6 | 4,21722 | 0,99999 | 0,00000 | 4,65304 | 0,99999 | 0,00000 | 5,08415 | 1,00000 | 0,00000 |
| 3,8 | 4,41722 | 1,00000 | 0,00000 | 4,85304 | 1,00000 | 0,00000 | — | — | — |

$$f = 2B^2; \quad H = \frac{A}{B}; \quad t^{**} = CB.$$

We will pursue now the physical meaning of the integral characteristics of the boundary layer (2.34). The values $A$, $B$ and $D$ are proportional respectively to the conventional displacement thicknesses, to the impulse loss and to the energy of the boundary layer. The value $F_0$ is proportional to the work of the tangential forces, and is equal to the energy converted into heat as a result of friction between the particles of a viscous fluid. The second derivative $f_0''(0)$ is proportional to the local friction stress. The values $f$ and $H$ are shape parameters, characterizing the fullness of the velocity profile across the boundary layer, and $t^{**}$ is a parameter depending on the intensity of suction of the fluid across a porous surface.

The data of the calculations of the functions $f_0$, $f_0'$ and $f_0''$ as a function of the dimensionless coordinate $\eta$ for different values of the parameter $C$ are shown in Table 5, and the graph (Fig. 12) shows the dimensionless local velocity across the boundary layer as a function of the coordinate which is proportional to the distance along the normal to the surface of the body.

Table 6 and Figure 13 show the values of the integral characteristics of the boundary layer $A$, $B$, $D$, $F_0$, $f_0''(0)$, $f$, $H$ and $t^{**}$. /58
The basic characteristics of the boundary layer change in full correlation to the physical processes, flowing with fluid suction from the boundary layer through the porous surface of the body.

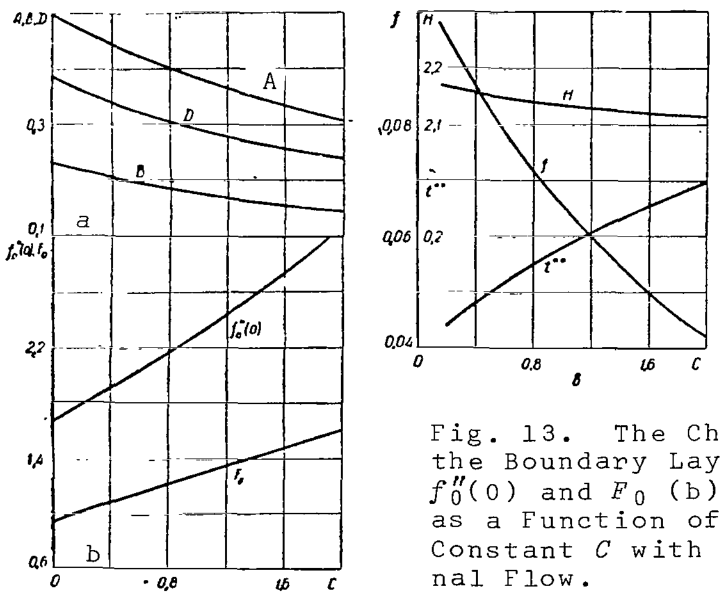



Fig. 13. The Characteristics of the Boundary Layer $A$, $B$ and $D$ (a), $f_0''(0)$ and $F_0$ (b) and $f$, $H$, $t^{**}$ (c) as a Function of the Value of the Constant $C$ with Exponential External Flow.

## The Solution for the General Case of a Boundary Layer in Series Form

For the arbitrary velocity distributions in the external flow and suction along a porous surface, several methods of calculating a laminar boundary layer were developed which in a certain sense can be considered precise. They can be classified as methods of solution based on the use of series expansion and methods based on finite difference calculus.

TABLE 6. VALUES OF THE INTEGRAL CHARACTERISTICS OF A LAMINAR BOUNDARY LAYER WITH EXPONENTIAL VELOCITY DISTRIBUTION ON THE OUTER BOUNDARY /59

| $\zeta$ | $A$ | $B$ | $F_0$ | $D$ | $f$ | $H$ | $t^{**}$ |
|---|---|---|---|---|---|---|---|
| 0,2 | 0,4763 | 0,2200 | 1,0166 | 0,3601 | 0,0965 | 2,1682 | 0,0439 |
| 0,4 | 0,4519 | 0,2092 | 1,0768 | 0,3433 | 0,0875 | 2,1600 | 0,0837 |
| 0,6 | 0,4290 | 0,1994 | 1,1400 | 0,3275 | 0,0795 | 2,1518 | 0,1196 |
| 0,8 | 0,4077 | 0,1901 | 1,2052 | 0,3123 | 0,0723 | 2,1445 | 0,1521 |
| 1,0 | 0,3878 | 0,18140 | 1,2731 | 0,2985 | 0,0658 | 2,1378 | 0,1814 |
| 1,4 | 0,3520 | 0,1656 | 1,4160 | 0,2728 | 0,0548 | 2,1261 | 0,2318 |
| 1,8 | 0,3287 | 0,1516 | 1,5676 | 0,25020 | 0,0459 | 2,1165 | 0,2729 |

In order to calculate the laminar boundary layer using series, which are applicable with power velocity distribution on the outer boundary of the layer, universal functions for the coefficients of these series were computed. Due to poor convergence, these series are used only to obtain initial values with the use of the finite difference method.

Gortler [65] - [67] proposed a new method, based on the representation of the solution in series form with high convergence and the introduction of the new dimensionless variables:

$$\xi = \frac{1}{\nu}\int_0^x U(x)\,dx; \qquad \eta = \frac{U(x)\,y}{\left[2\nu \int_0^x U(x)\,dx\right]^{\frac{1}{2}}}. \tag{2.35}$$

We now introduce the function

$$f_0(\xi,\eta) = \frac{\psi(x,y)}{\nu\sqrt{2\xi}}, \tag{2.36}$$

where $\psi(x,y)$ is the current function. Thus for the velocity components we obtain

$$u(x,y) = \frac{\partial f_0}{\partial \eta}(\xi,\eta)\,U(x); \tag{2.37}$$

$$v(x, y) = -\frac{1}{\sqrt{2\xi}}\left[f_0(\xi, \eta) + 2\xi\frac{\partial f_0}{\partial \xi}(\xi, \eta) + (\beta(\xi) - \right.$$

$$\left. - 1)\,\eta\frac{\partial f_0}{\partial \eta}(\xi, \eta)\right] U(x). \qquad (2.38)$$ /60

and we convert the system of equations for the boundary layer to its final form

$$\frac{\partial^3 f_0}{\partial \eta^3} + f_0\frac{\partial^2 f_0}{\partial \eta^2} + \beta(\xi)\left[1 - \left(\frac{\partial f_0}{\partial \eta}\right)^2\right] =$$

$$= 2\xi\left[\frac{\partial f_0}{\partial \eta}\cdot\frac{\partial^2 f_0}{\partial \xi \partial \eta} - \frac{\partial f_0}{\partial \xi}\cdot\frac{\partial^2 f_0}{\partial \eta^2}\right], \qquad (2.39)$$

where the function

$$\beta(\xi) = 2\xi\frac{d}{d\xi}\ln\left[\frac{U(x)}{U_0}\right] = \frac{2\frac{dU}{dx}}{U^2(x)}\int_0^x U(x)\,dx, \qquad (2.40)$$

and the boundary conditions are the following:

$$\frac{\partial f_0}{\partial \eta}(\xi, \eta) = 0; \quad \lim\frac{\partial f_0}{\partial \eta}(\xi, \eta) = 1 \text{ with } \eta \to \infty;$$

$$f_0(\xi, 0) + 2\xi\frac{\partial f_0}{\partial \xi}(\xi, 0) = \gamma(\xi). \qquad (2.41)$$

The function

$$\gamma(\xi) = -\frac{v_0(x)}{U(x)}\left[\frac{2}{\nu}\int_0^x U(x)\,dx\right]^{\frac{1}{2}} \qquad (2.42)$$

takes into account the presence of suction of the boundary layer.

We will assume that functions (2.40) and (2.42) can be expanded to a series over $\xi$, converging in the interval of interest to us, and we will represent these functions in the form of power series

$$\beta(\xi) = \sum_{k=0}^{\infty}\beta_k\xi^k; \qquad \gamma(\xi) = \sum_{k=0}^{\infty}(2k+1)\gamma_k\xi^k. \qquad (2.43)$$

In addition, we assume that

$$U(x) = x^m [s_0 + s_1 x^{m+1} + s_2 x^{2(m+1)} + \ldots], \tag{2.44}$$ /61

where $m = \beta_0/(2 - \beta_0)$ and $s_0 \neq 0$. The expression $m = 0$ ($\beta_0 = 0$) refers to the flow near the critical point of the plate, and $m = 1$ ($\beta_0 = 1$) refers to the streamlining of the rounded leading edge.

The distribution of the suction rate can also be expanded to the power series

$$v_0(x) = x^{\frac{m-1}{2}} \sum_{k=0}^{\infty} \sigma_k x^{k(m+1)}. \tag{2.45}$$

We will seek the solution to equation (2.39) in the form

$$f_0(\xi, \eta) = \sum_{k=0}^{\infty} f_{0_k}(\eta) \xi^k. \tag{2.46}$$

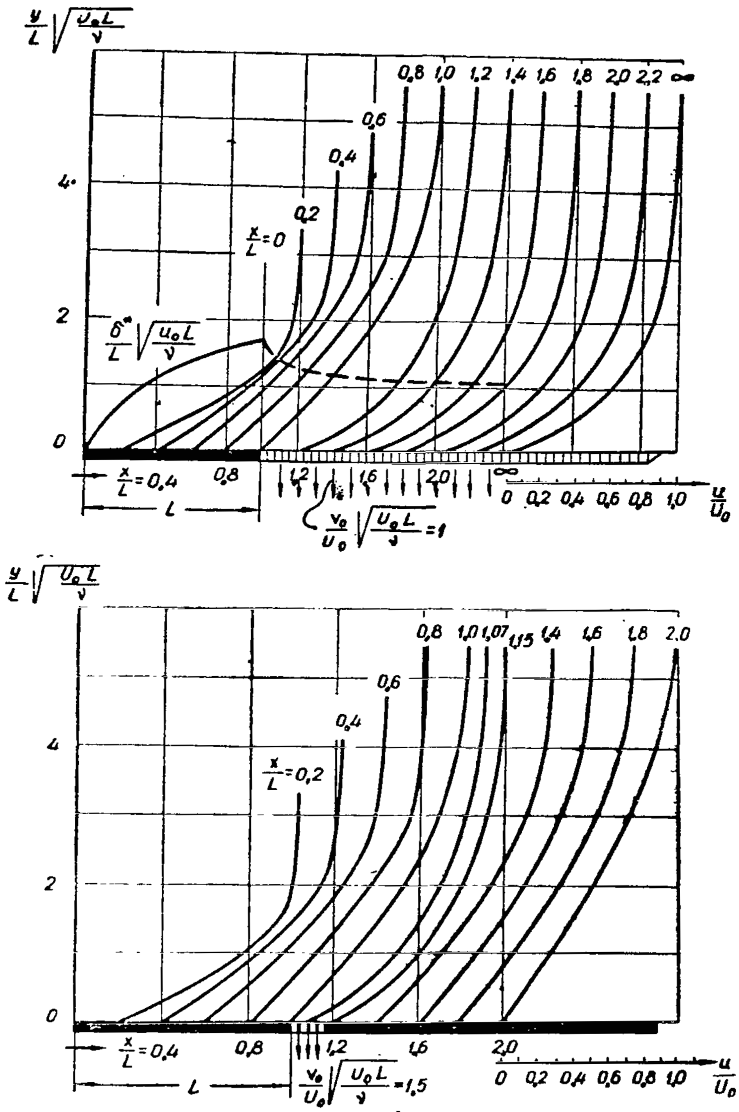
The determination of the coefficients of the functions $f_{0_1}(\eta)$ can be reduced to the computation of auxiliary functions. However, the number of auxiliary functions increases very rapidly with an increase in the order of approximation. Thus, for example, for fourth order there will be 38 auxiliary functions.

For the practical use of boundary layer suction, the case is interesting when suction begins at a certain distance $x_0$ from the leading edge of the body. Then the suction distribution is a continuous function of a coordinate along the surface of the body.

W. Rheinboldt [97] proposed a solution to this problem for the general case of velocity distribution along the outer boundary of the boundary layer in the form of the expansions

$$\left.\begin{aligned} u(y) &= \sum_{n=1}^{\infty} a_n y^n; \\ U(x) &= \sum_{n=0}^{\infty} U_n (x - x_0)^n; \\ v_0(x) &= \sum_{n=0}^{\infty} v_n (x - x_0)^n \end{aligned}\right\} \tag{2.47}$$

(in the last expansion $v_0 \neq 0$).




b

/62

Fig. 14. Flow With Uniform Suction of a Laminar Boundary Layer on a Plate (a) and Through a Slot (b) (According to Rheinboldt).

When expansions (2.47) are applied at point $x = x_0$; $y = 0$ there /63 is a singularity, which we can investigate by the transformation of coordinates

$$\left.\begin{aligned} \sigma &= \sqrt[N]{x - x_0}; \\ \tau &= \frac{y}{N\sqrt[N]{x - x_0}}; \\ \psi(x, y) &= \sigma^{N-1}\varphi(\sigma, \tau). \end{aligned}\right\} \qquad (2.48)$$

The current function, which for large values of $x$ tends to an asymptotic solution, is defined in these new variables.

As an example, Rheinboldt examined a flat plate with uniform suction of a boundary layer downstream for two variants: beginning with a given point $x_0$ (Fig. 14a) and for uniform suction between two points $x_0$ and $x_1$ (Fig. 14b).

## Use of the General Method of Finite Differences

The second trend in the development of precise methods for integrating the equations for a laminar boundary layer is the application of the general method of finite differences [131]. The method of finite differences as applied to the equations of a boundary was developed by Schroder [104] and Görtler [64]. We will now examine Görtler's method with the refinements for solution near the surface, which were proposed by Witting [124].

We introduce the dimensionless variables

$$\bar{x} = \frac{x}{L}; \quad \bar{y} = \frac{y\sqrt{\mathrm{Re}}}{L}; \quad \bar{p} = \frac{p}{\frac{1}{2}\varrho U_0^2};$$
$$\bar{u} = \frac{u}{U_0}; \quad \bar{v} = \frac{v\sqrt{\mathrm{Re}}}{U_0}; \quad \bar{U} = \frac{U}{U_0},$$

where $U_0$ is the characteristic velocity and $L$ is the characteristic length.

After the exclusion of the value $v$, with the help of the equation of continuity for a plane boundary layer, we can write equation (2.7) in the presence of suction in the form

$$u\frac{\partial u}{\partial x} + \left(v_0 - \int_0^y \frac{\partial u}{\partial x}\,dy\right)\frac{du}{dy} = U\frac{dU}{dx} + \frac{\partial^2 u}{\partial y^2}. \quad (2.49)$$

For a point with coordinates $x_i$, $y_k$ (Fig. 15) /64

$$x_i = x_0 + ih; \quad i = 0, \pm 1, \pm 2; \ldots; h > 0;$$
$$y_k = kl; \quad k = 0, 1, 2; \ldots; l > 0;$$
$$u_{i,k} = u(x_i, y_k).$$

We will express the derivatives as finite differences:

$$\left.\begin{aligned} \frac{\partial u}{\partial x}(x_i, y_k) &= \frac{\Delta_{i,k}}{2h}; \\ \frac{\partial u}{\partial y}(x_i, y_k) &= \frac{\nabla_{i,k}}{2l}\ (k \geqslant 1); \\ \frac{\partial^2 u}{\partial y^2}(x_i, y_k) &= \frac{\nabla^2_{i,k}}{4l^2}\ (k \geqslant 2), \end{aligned}\right| \tag{2.50}$$

where

$$\Delta_{i,k} = u_{i+1,k} - u_{i-1,k};$$
$$\nabla_{i,k} = u_{i,k+1} - u_{i,k-1};$$
$$\nabla^2_{i,k} = \nabla_{i,k+1} - \nabla_{i,k-1}.$$

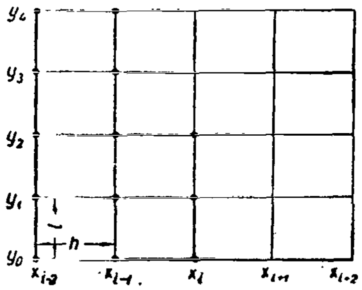



Fig. 15. A Grid for Computing Finite Differences.

Using the trapezoidal rule for the integral in equation (2.49) we obtain

$$\Delta_{i,k} - \left\{4hU_iU_i' + \frac{h}{l^3}\nabla^2_{i,k} + \nabla_{i,k}\left(\sum_{r=1}^{k-1}\Delta_{i,r} - \frac{2h}{l}v_{i,0}\right)\right\} \times \\ \times \frac{1}{2(u_{i,k} - h\nabla_{i,k})}, \tag{2.51}$$

where

$$U' = \frac{dU}{dx}; \qquad k \geqslant 1.$$

The differences $\nabla_{i,0}$, $\nabla^2_{i,0}$, $\nabla^2_{i,1}$ near the wall can be computed in the following form. From equation (2.49) with $y = 0$, we have

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = v\left(\frac{\partial u}{\partial y}\right)_0 + \frac{\partial p}{\partial x} = v_0\left(\frac{\partial u}{\partial y}\right)_0 - UU'. \tag{2.52}$$

Differentiating equation (2.49) we obtain /65

$$\left(\frac{\partial^3 u}{\partial y^3}\right)_0 = v_0\left(\frac{\partial^2 u}{\partial y^2}\right)_0. \tag{2.53}$$

We can write the approximate relationship near the wall for the three points $y_1$, $y_2$ and $y_3$ as:

$$\frac{\partial^2 u}{\partial y^2} = \left\{v_0\left(\frac{\partial u}{\partial y}\right)_0 - UU'\right\} + v_0\left(\frac{\partial^2 u}{\partial y^2}\right)_0 y + c(x)y^2 + d(x)y^3$$

and thus we obtain three equations for determining $c(x)$, $d(x)$ and $\nabla^2_{i,1}$, from which it follows that

$$\nabla^2_{i,1} = \nabla^2_{i,0}\left(\frac{11}{18} + \frac{1}{3} lv_{i,0}\right) + \frac{1}{2}\nabla^2_{i,2} - \frac{1}{9}\nabla^2_{i,3}. \tag{2.54}$$

In addition

$$\nabla^2_{i,0} = -4l^2UU' + v_{i,0}2l\nabla_{i,0}.$$

Denoting

$$\nabla_{i,0} = \nabla_{i,2} - \nabla_{i,1},$$

after simple transformations we obtain the relationship

$$\nabla_{i,0} = \left\{\frac{1}{1 + v_{i,0}2l\left(\frac{11}{18} + \frac{lv_{i,0}}{3}\right)}\right\}\left[4l^2UU'\left(\frac{11}{18} + \frac{lv_{i,0}}{3}\right) + \nabla_{i,2} - \frac{1}{2}\nabla^2_{i,2} + \frac{1}{9}\nabla^2_{i,3}\right]. \tag{2.55}$$

This relationship is the generalized Görtler formula for a permeable surface in the presence of suction.

Rheinboldt [97] found that to obtain more stability in solution it was necessary to use the finite differences

$$\frac{\partial u}{\partial x}(x_i, y_k) = \frac{\nabla_{i,k}}{h} = \frac{1}{h}(u_{i\pm 1,k} - u_{i,k}). \tag{2.56}$$

A somewhat different approach to the application of the finite difference method for solving the equations of an incompressible laminar boundary layer in the presence of suction was proposed by Smith and Clutter [107]. According to the above-mentioned reference, the system of equations for the boundary layer and the equation of discontinuity (2.7) can be reduced to one third-order equation rel- /66
ative to the current function $\psi$ by using the relationships

$$u = \frac{\partial \psi}{\partial y}; \tag{2.57}$$

$$v = -\frac{\partial \psi}{\partial x}. \tag{2.58}$$

As a result we obtain the equation

$$\frac{\partial\psi}{\partial y}\cdot\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\cdot\frac{\partial^2\psi}{\partial y^2}=U\frac{\partial U}{\partial x}+\nu\frac{\partial^3\psi}{\partial y^3}, \tag{2.59}$$

while the boundary conditions will be

$$\left(\frac{\partial\psi}{\partial y}\right)=0; \qquad \frac{\partial\psi}{\partial x}=-v_0 \text{ with } y=0;$$

$$\frac{\partial\psi}{\partial y}\to U(x) \qquad \text{with } y\to\infty. \tag{2.60}$$

Using the new variables which were first introduced by Falkner and Skan

$$\psi=(U\nu x)^{\frac{1}{2}}f_0(x,\eta);$$
$$\eta=(U/\nu x)^{\frac{1}{2}}y, \tag{2.61}$$

we convert equation (2.7) to the form

$$f_0'''=-\left(\frac{M+1}{2}\right)f_0f_0''+M(f_0'^2-1)+x\left[f_0'\frac{\partial f_0'}{\partial x}-f_0''\frac{\partial f_0}{\partial x}\right]. \tag{2.62}$$

The parameter $M=(\frac{x}{U})(\frac{dU}{dx})$ takes into account the velocity gradient on the outer boundary of the boundary layer. The boundary conditions will be

$$f_0'=0; \qquad f_0=-(\nu Ux)^{-\frac{1}{2}}\int_0^x v_0(\xi)\,d\xi \text{ with } \eta=0;$$
$$f_0\to 1 \text{ with } \eta\to\infty, \tag{2.63}$$

where $\xi$ is the integration variable.

Equation (2.62) has important advantages in comparison to the other possible forms of the equations for the boundary layer, which, as a rule, have a natural singularity at the point $x = 0$. In addition, to solve them we must assign an initial profile. This profile is usually assigned at a certain distance from the leading edge.

Equation (2.62) does not contain singularities at the initial point. In this equation, with the condition that the derivatives of $f_0$ are finite, the expression in brackets, containing the derivatives over $x$, becomes zero, and thus has a self-similar solution. The form of this equation is such that the expression in brackets represents a correction to the self-similar solutions. In this case /67 the boundary condition on the outer boundary of the boundary layer is somewhat simplified.

The basic idea of this solution consists in the replacement of the derivative over $x$ by finite differences in order to reduce the equation in partial derivatives to an ordinary differential equation. As a result of thorough investigations into the use of two-point, three-point and four-point schemes, Smith and Clutter [107] decided on a three-point scheme, characterized by reasonable precision and the necessary stability.

In equation (2.62) both derivatives over $x$ are replaced by the Lagrange finite difference formula for three points. Let $m$ denote the order of the derivative, then

$$\left(\frac{\partial f^{(m)}}{\partial x}\right)_n = \left[\frac{1}{x_n - x_{n-1}} + \frac{1}{x_n - x_{n-2}}\right] f_n^{(m)} - \left[\frac{x_n - x_{n-2}}{(x_n - x_{n-1})(x_{n-1} - x_{n-2})}\right] f_{n-1}^{(m)} + \left[\frac{x_n - x_{n-1}}{(x_n - x_{n-2})(x_{n-1} - x_{n-2})}\right] f_{n-2}^{(m)}. \tag{2.64}$$

The error in formula (2.64) is of the order $\left[\frac{(\Delta x)^2}{3}\right]\left(\frac{\partial^2 f^{(m)}}{\partial x^3}\right)$.

The difficulties in solving equation (2.62) numerically are caused both by its nonlinearity and by the fact that one of the boundary conditions was assigned with $\eta = \infty$. With integration we use the method of successive approximations employing various extrapolation formulas.

The problem with the general methods, which are based on using /68 the finite difference scheme, is that they are very unwieldy. However, if we make the calculations on a computer, this problem is eliminated.

## Possible Solutions for Certain Classes of Problems

The general theory of self-similar solutions to the equations for a boundary layer in the plane case was developed by Holstein [76], Mangler [85] and Wuest [131]. In order to make the velocity profile self-similar, the following conditions must be satisfied:

$$\frac{u(x, y)}{U(x)} = \frac{d\sigma(\eta)}{d\eta} = \sigma'(\eta), \tag{2.65}$$

where

$$\eta = \sqrt{\mathrm{Re}}\, g(x)\, y.$$

Integrating the equation of continuity of system (2.7)

$$\sqrt{\mathrm{Re}}\, v = \sigma \frac{d}{dx}\left(\frac{U}{g}\right) + \sigma' \eta \left(\frac{U}{g}\right) \tag{2.66}$$

and substituting this relationship into the Prandtl equation, we obtain an ordinary differential equation

$$\sigma''' + a_1 \sigma\sigma'' + a_2 (1 - \sigma'^2) = 0, \tag{2.67}$$

where

$$a_1 = \frac{1}{g}\left\{\frac{d\left(\frac{U}{g}\right)}{dx}\right\};$$

$$a_2 = \frac{1}{g^2} \cdot \frac{dU}{dx}.$$

From equation (2.66) we find the suction distribution along the surface of the body

$$\sqrt{\mathrm{Re}}\, v = \sigma_0 \left\{\frac{d\left(\frac{U}{g}\right)}{dx}\right\} = a_1 \sigma_0 g, \tag{2.68}$$

since

$$\sigma'(0) = 0; \quad \sigma(0) = \mathrm{const}.$$

With supplementary conditions we can find several variants of the solutions. Table 7 [131] shows the possible self-similar solutions. /69

Let us now examine three-dimensional self-similar boundary layers in the presence of suction according to the results of Wuest's investigations [131].

1. Self-similarity along a surface. In this case the velocity profiles on the entire surface are affine and the velocity profile corresponds to the expression

$$\frac{u(x, y, z)}{U^*(x, y)} = \sigma'(\xi); \qquad \frac{v(x, y, z)}{V^*(x, y)} = \tau'(\xi). \tag{2.69}$$

where $\xi = \sqrt{Re}\ g(x,y)z$; $x$ and $y$ are the coordinates along the surface; $z$ is the coordinate normal to the surface; $U^*$ and $V^*$ are the scale factors for velocity. If the velocities $U$ and $V$ on the outer boundary of the boundary layer do not become equal to zero, then we can take $U^* = U$ and $V^* = V$. Substituting relationship (2.69) into the system of equations

$$\left.\begin{aligned}
&u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\varrho}\cdot\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2};\\
&u\frac{\partial v}{\partial x} + v\frac{\partial v}{dy} + w\frac{\partial v}{\partial z} = -\frac{1}{\varrho}\cdot\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2};\\
&\frac{\partial u}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial w}{\partial z} = 0,
\end{aligned}\right\} \tag{2.70}$$

we obtain the system of ordinary differential equations

$$\begin{aligned}
&\sigma''' + \sigma''(a_1\sigma + b_1\tau) + a_2(1-\sigma'^2) + a_3(1-\sigma'\tau') = 0;\\
&\tau''' + \tau''(a_1\sigma + b_1\tau) + b_2(1-\tau'^2) + b_3(1-\sigma'\tau') = 0,
\end{aligned} \tag{2.71}$$

where

$$a_1 = \frac{1}{g}\cdot\frac{\partial}{\partial x}\left(\frac{U^*}{g}\right);\qquad a_2 = \frac{1}{g^2}\cdot\frac{\partial U^*}{\partial x};\qquad a_3 = \frac{V^*}{g^2}\cdot\frac{1}{U^*}\cdot\frac{\partial U^*}{\partial y};$$

$$b_1 = \frac{1}{g}\cdot\frac{\partial}{\partial y}\left(\frac{V^*}{g}\right);\qquad b_2 = \frac{1}{g^2}\cdot\frac{\partial V^*}{\partial y};$$

$$b_3 = \frac{U^*}{g^2}\cdot\frac{1}{V^*}\cdot\frac{\partial V^*}{\partial x}.$$

In this case the boundary conditions on the wall ($z = 0$) will have the form

$$\sigma'(0) = \tau'(0) = 0;$$

$$\sigma(0) = \sigma_0 = \text{const};\quad \tau(0) = \tau_0 = \text{const}.$$

The conditions on the outer boundary of the boundary layer are a function of the choice of $U^*$ and $V^*$. If $U^* = U$ and $V^* = V$, they will correspond to the expression $\sigma'(\infty) = \tau'(\infty) = 1$. To satisfy the boundary conditions on the wall the suction rate must satisfy the equation

$$\begin{aligned}
&\sqrt{Re}\, w_0(x, y) = -g(a_1\sigma_0 +\\
&+ b_1\tau_0) = -\text{const}\cdot g.
\end{aligned} \tag{2.72}$$

TABLE 7. SELF-SIMILAR SOLUTIONS TO THE EQUATIONS FOR A PLANE BOUNDARY LAYER IN THE PRESENCE OF SUCTION. /70

| Type of Flow | $U$ | $a_1$ | $a_2$ | $\varepsilon$ | $a_0$ | Source |
|---|---|---|---|---|---|---|
| A Flat Plate | $U=\beta=\mathrm{const}$ | 1 | 0 | $\sqrt{\frac{\beta}{2x}}$ | $\sim x^{-\frac{1}{2}}$ | [102], [111], [119], [62] |
| The Critical Point | $U=\beta x$ | 1 | 1 | $\sqrt{\beta}$ | const | [102] |
| Flow in the Diffusion Section | $U=\beta x^{a}$ | 1 | $-\infty<a_2\leqslant 0$ | | | [85] for $a_2=0.1$ and $a_2=-0.25$ |
| Flow in the Wedges | | 1 | $0\leqslant a_2<2$ | $\frac{\sqrt{\alpha\beta}}{a_2}x^{\frac{\alpha-1}{2}}$ | $\sim x^{\frac{\alpha-1}{4}}$ | [79] for $-0.5<a_2<0$ |
| Flow in the Convergence Section | | 1 | $2<a_2\leqslant\infty$ | | | [98] $a_2=0.2$ |
| Flow in the Diffusion Section | $\alpha=\frac{1}{2\left(\frac{a_1}{a_2}\right)-1}$ | −1 | $-\infty<a_2<-2$ | | | [83] |
| Backflow in the Wedges | | −1 | $-2<a_2\leqslant 0$ | | | |
| Flow in the Convergence Section | | −1 | $0\leqslant a_2\leqslant\infty$ | | | |
| Flow in the Convergence Section | $U=\beta e^{2x}$ | 1 | 2 | $\sqrt{\beta e^{x}}$ | $e^{x}$ | |
| Flow in the Diffusion Section | $U=\beta e^{-2x}$ | −1 | −2 | $\sqrt{\beta e^{-x}}$ | $e^{-x}$ | |
| Flow from the Drain | $U=\frac{\beta}{x}$ | 0 | 1 | $\frac{\sqrt{\beta}}{x}$ | $\sim\frac{1}{x}$ | |
| Flow from the Source | $U=\frac{\beta}{x}$ | 0 | −1 | | | |

These results are analogous to a plane boundary layer. However, for a spatial boundary layer, $g$ is a function in two variables. Possible solutions for the spatial boundary layer in the presence of suction and [self-similarity along a surface] are shown in Table 8, while for cases 7 and 8, assuming $m_1 = m_2 = m$ or $n_1 = n_2 = n$, we can obtain the additional solution /71

$$U = A(px + gy)^n;$$

$$V = B(px + gy)^n;$$

$$g = C(px + gy)^{\frac{n-1}{2}}.$$

TABLE 8. THREE-DIMENSIONAL BOUNDARY LAYER IN THE PRESENCE OF SUCTION WITH SELF-SIMILARITY ALONG A SURFACE.

| Parameters | Case 1 | Case 2 | Case 3 | Case 4 | Case 5 | Case 6 | Case 7 | Case 8 |
|---|---|---|---|---|---|---|---|---|
| $U$ | $Ae^{px+gy}$ | $Ax^{m}y^{n-1}$ | $Ax^{m}e^{gy}$ | $Ay^{n-1}e^{px}$ | $Ae^{p_1x}$ | $Ae^{g_1y}$ | $Ax^{m_1}$ | $Ay^{n_1}$ |
| $V$ | $Be^{px+gy}$ | $Bx^{m-1}y^{n}$ | $Bx^{m-1}e^{gy}$ | $By^{n}e^{px}$ | $Be^{p_2x}$ | $Be^{g_2y}$ | $Bx^{m_2}$ | $By^{n_2}$ |
| $g\sim w_0$ | $Ce^{\frac{(px+gy)}{2}}$ | $Cx^{\frac{(m-1)}{2}}y$ | $Cx^{\frac{(m-1)}{2}}e^{\frac{gy}{2}}$ | $Cy^{\frac{(n-1)}{2}}e^{\frac{px}{2}}$ | $Ce^{\frac{p_1x}{2}}$ | $Ce^{g_2\frac{y}{2}}$ | $Cx^{\frac{(m_1-1)}{2}}$ | $Cy^{\frac{(n_2-1)}{2}}$ |

2. Self-Similarity Along a Line. In this case the velocity profiles are affine along the line $\chi(x,y)$ = const. Thus for the velocity components $u$ and $v$

$$\frac{u(x, y, z)}{U^*(x, y)}=\frac{\partial\sigma(\chi, \zeta)}{\partial\xi}; \qquad \frac{v(x, y, z)}{V^*(\chi, y)}=\frac{\partial\tau(\chi, \zeta)}{\partial\xi}, \tag{2.73}$$ /72

where

$$\xi=\sqrt{\mathrm{Re}}\, g(x, y) z.$$

Substituting this relationship into the first equation of system (2.7), we obtain the following system of differential equations in partial derivatives with the two dependent variables $\chi$ and $\xi$.

$$\begin{gathered}\sigma_{\zeta\zeta\zeta}+\sigma_{\zeta\zeta}(a_1\sigma+b_1\tau)+a_2(1-a_\zeta^2)+a_3(1-\sigma_\zeta\tau_\zeta)+\\ +\sigma_{\zeta\zeta}(a_4\sigma_\chi+b_4\tau_\chi)-a_4\sigma_\zeta\sigma_{\zeta\chi}+b_4\tau_\zeta\sigma_{\zeta\chi}=0;\\ \tau_{\zeta\zeta\zeta}+\tau_{\zeta\zeta}(a_1\sigma+b_1\tau)+b_2(1-\tau_\zeta^2)+b_3(1-\sigma_\zeta\tau_\zeta)-\\ -a_4\sigma_\zeta\tau_{\zeta\chi}-b_4\tau_\zeta\tau_{\zeta\chi}=0.\end{gathered} \tag{2.74}$$

Here $a_1$, $a_2$, $a_3$; $b_1$, $b_2$, $b_3$ have the previous values, but they are no longer constants; rather they are functions of $\chi$.

In addition

$$a_4=\frac{U^*\chi_x}{g^2}; \qquad b_4=\frac{(V^*\chi_y)}{g^2}. \tag{2.75}$$

The suction law must satisfy the equation

$$-\sqrt{\mathrm{Re}}\, w_0=g(a_1\sigma_0+b_1\tau_0+a_4\sigma_{0\chi}+b_4\tau_{0\chi}). \tag{2.76}$$

The expression in parenthesis is a function only of $\chi$.

Numerical calculations of a spatial boundary layer in the presence of suction and linear self-similarity were not made until recently.

## A Boundary Layer With Intense Suction

Let us now examine the case of intense suction of a fluid from the boundary layer when the suction rate is high. Then, as Pretsch explained [95], [96], the velocity profile in the boundary layer with an arbitrarily assigned suction rate $v_0(x)$ and velocity distribution along the outer boundary of the layer $U(x)$ is reasonably well approximated by the asymptotic suction profile (2.3).

If /73

$$\psi = \int_0^x v_0(x)\,dx + \frac{\nu U(x)}{v_0(x)} f_0(x, \zeta), \tag{2.77}$$

where

$$\zeta = \frac{v_0 y}{\nu},$$

then the velocity components in the boundary layer are

$$u = U\frac{\partial f_0}{\partial \zeta}; \quad v = -v_0 - \left(\frac{\nu}{v_0^2}\right)\left[\frac{dU}{dx} v_0 f_0 + U v_0 \frac{\partial f_0}{\partial x} + \right.$$
$$\left. + U\frac{dv_0}{dx}\left(\zeta\frac{\partial f_0}{\partial \zeta} - f_0\right)\right]. \tag{2.78}$$

We can convert the equation of fluid motion (the first equation of system (2.1)) to the form

$$\frac{\partial^3 f_0}{\partial \zeta^3} + \frac{\partial^2 f_0}{\partial \zeta^2} + \left(\frac{\nu}{v_0^3}\right)\left\{\frac{dU}{dx} v_0\left[1 - \left(\frac{\partial f_0}{\partial \zeta}\right)^2 + f_0\frac{\partial^2 f_0}{\partial \zeta^2}\right] + \right.$$
$$\left. + U v_0\left[\frac{\partial f_0}{\partial x}\frac{\partial^2 f_0}{\partial \zeta^2} - \frac{\partial^2 f_0}{\partial x \partial \zeta}\frac{\partial f_0}{\partial \zeta}\right] - U\frac{\partial v_0}{\partial x} f_0 \frac{\partial^2 f_0}{\partial \zeta^2}\right\} = 0, \tag{2.79}$$

and the boundary conditions will be

$$f_0 = C; \quad \frac{\partial f_0}{\partial \zeta} = 0 \text{ with } \zeta = 0;$$
$$\frac{\partial f_0}{\partial \zeta} \to 1 \qquad \text{with } \zeta \to \infty. \tag{2.80}$$

When the value of $v_0$ is very high, equation (2.79) can be simplified considerably and takes the form

$$\frac{\partial^3 f_0}{\partial \zeta^3} + \frac{\partial^2 f_0}{\partial \zeta^2} = 0. \tag{2.81}$$

The solution to equation (2.81) when the boundary conditions are satisfied is

$$\frac{\partial f_0}{\partial \zeta} = 1 - e^{-\zeta}. \tag{2.82}$$

Comparing expression (2.82) with the solution to (2.3), which we /74
obtained earlier, we arrive at the conclusion that the limiting shape of the velocity distribution across the boundary layer in the presence of intense suction is an asymptotic profile. In particular, the characteristic thicknesses of the boundary layer are inversely proportional to the suction rate $v_0$ and are not a function of the local velocity on the outer boundary of the layer.

Analogous reasons allowed Watson [120] to obtain an analogous expression for the velocity profile in the case when the suction rate $v_0$ is high. Approximately, this expression has the form

$$f_0(x,\zeta)=f_{0,0}+\left(\frac{\nu}{v_0^3}\right)\left(\frac{\partial U}{dx}v_0f_{1,0}+U\frac{dv_0}{dx}f_{0,1}\right)+\left(\frac{\nu^2}{v_0^6}\right)\left[U\frac{dU}{dx}v_0^2f_{2,0}+\right.$$
$$\left.+\left(\frac{dU}{dx}\right)^2v_0^2f_{11,0}+U\frac{dU}{dx}v_0\frac{dv_0}{dx}f_{1,1}+\ldots+U^2v_0\frac{d^2v_0}{dx^2}f_{0,2}+U^2\left(\frac{dv_0}{dx}\right)^2f_{0,11}\right]+\ldots, \tag{2.83}$$

where $f_{0,0}, f_{1,0}, f_{0,1},\ldots$ are functions in $\zeta$.

After substituting expression (2.83) into equation (2.63) and satisfying the boundary conditions (2.65), we can compute the functions $f_{0,0}, f_{1,0}, f_{0,1},\ldots$. Then for the velocity profile we obtain the expression

$$\frac{u}{U}=f'_{0,0}+\left(\frac{\nu}{v_0^3}\right)\left(\frac{dU}{dx}v_0f'_{1,0}+U\frac{dv_0}{dx}f'_{0,1}\right)+$$
$$+\left(\frac{\nu^2}{v_0^6}\right)\left[U\frac{d^2U}{dx^2}\frac{d^2v_0}{dx^2}f'_{2,0}+\left(\frac{dU}{dx}\right)^2v_0^2f'_{11,0}+U\frac{dU}{dx}v_0\frac{dv_0}{dx}f'_{1,1}+\right. \tag{2.84}$$
$$\left.+U^2v_0\frac{d^2v_0}{dx}f'_{0,2}+U^2\left(\frac{dv_0}{dx}\right)^2f'_{0,11}\right]+\ldots,$$

where

$$f_{0,0}=1-e^{-\zeta};$$
$$f'_{1,0}=\left(\frac{1}{2}\zeta^2+2\zeta\right)e^{-\zeta};$$
$$f'_{0,1}=-\left(\frac{1}{2}\zeta^2+\frac{1}{2}\right)e^{-\zeta}+\frac{1}{2}e^{-2\zeta}; \tag{2.85}$$

/75

$$f'_{2,0}=-\left(\frac{1}{6}\zeta^3+\frac{3}{2}\zeta^2+\frac{9}{4}\right)e^{-\zeta}+\left(\frac{1}{2}\zeta+\frac{9}{4}\right)e^{-2\zeta};$$
$$f'_{11,0}=-\left(\frac{1}{8}\zeta^4+\zeta^3+\frac{7}{2}\zeta^2+6\zeta+\frac{1}{2}\right)e^{-\zeta}+\frac{1}{2}e^{-2\zeta};$$

$$f'_{1,0}=\left(\frac{1}{4}\zeta^4+\frac{3}{2}\zeta^3+\frac{23}{4}\zeta^2+3\zeta+\frac{21}{2}\right)e^{-\zeta}-\left(\frac{1}{2}\zeta^2+\frac{9}{2}\zeta+\frac{21}{2}\right)e^{-2\zeta};$$

$$f'_{0,2}=\left(\frac{1}{6}\zeta^3+\frac{1}{2}\zeta^2+\frac{1}{4}\zeta+\frac{25}{24}\right)e^{-\zeta}-\left(\frac{1}{2}\zeta+1\right)e^{-2\zeta}-\frac{1}{24}e^{-3\zeta};$$

$$f_{0,11}=-\left(\frac{1}{8}\zeta^4+\frac{1}{2}\zeta^3+\frac{9}{4}\zeta^2+\frac{1}{2}\zeta+\frac{65}{12}\right)e^{-\zeta}+$$
$$+\left(\frac{1}{2}\zeta^2+\frac{5}{2}\zeta+\frac{11}{2}\right)e^{-2\zeta}-\frac{1}{12}e^{-3\zeta}.$$

The characteristics of the boundary layer can be computed using series (2.84). In particular, the dimensionless value of the coefficient of local friction is

$$\frac{\tau_0}{\varrho v_0 U}=1+\left(\frac{\nu}{v_0^3}\right)\left(2\frac{dU}{dx}v_0-\frac{1}{2}U\frac{dv_0}{dx}\right)+$$
$$+\left(\frac{\nu^2}{v_0^6}\right)\left[-\frac{7}{4}U\frac{d^2U}{dx^2}v_0^2-\frac{13}{2}\left(\frac{dU}{dx}\right)^2v_0^2+9U\frac{dU}{dx}v_0\frac{dv_0}{dx}+\right. \tag{2.86}$$
$$\left.+\frac{5}{6}U^2v_0\frac{d^2v_0}{dx^2}-\frac{10}{3}U^2\left(\frac{dv_0}{dx}\right)^2\right]+\ldots;$$

the dimensionless displacement thickness is

$$\frac{v_0\delta^*}{\nu}=1+\left(\frac{\nu}{v_0^3}\right)\left[-3\frac{dU}{dx}v_0+\frac{5}{4}U\frac{dv_0}{dx}\right]+\left(\frac{\nu^2}{v_0^6}\right)\left[5U\frac{d^2U}{dx^2}v_0^2+\right.$$
$$\left.+\frac{89}{4}\left(\frac{dU}{dx}\right)^2v_0^2-\frac{67}{2}U\frac{dU}{dx}v_0\frac{dv_0}{dx}-\frac{191}{72}U^2v_0\frac{d^2v_0}{dx^2}+\frac{233}{18}U^2\left(\frac{dv_0}{dx}\right)^2\right]+\ldots; \tag{2.87}$$

and the dimensionless thickness of impulse loss is

$$\frac{v_0\delta^{**}}{\nu}=\frac{1}{2}+\left(\frac{\nu}{v_0^3}\right)\left[-\frac{7}{4}\frac{dU}{dx}v_0+\frac{5}{6}U\frac{dv_0}{dx}\right]+$$
$$+\left(\frac{\nu^2}{v_0^6}\right)\left[\frac{251}{72}U\frac{d^2U}{dx^2}v_0^2+\frac{347}{24}\left(\frac{dU}{dx}\right)^2v_0^2-\right. \tag{2.88}$$
$$\left.-\frac{569}{24}U\frac{dU}{dx}v_0\frac{dv_0}{dx}-\frac{275}{144}U^2v_0\frac{d^2v_0}{dx^2}+\frac{1375}{144}U^2\left(\frac{dv_0}{dx}\right)^2\right]+\ldots.$$

# CHAPTER 3

## APPROXIMATE METHODS OF CALCULATION FOR A PLANE BOUNDARY LAYER /77

### The Use of Impulse Relationships

For an approximate calculation of a laminar boundary layer with an arbitrary velocity distribtuion along the outer boundary of the layer, we can use either the integral impulse relationship (1.34) or the integral relationships for impulses and energy (1.37) together. The essence of these methods is that instead of the actual velocity profiles in the cross sections of the boundary layer, we use an approximate one-parameter set.

Varying the parameter, called a shape factor, allows us to create a variety of profile shapes necessary for an approximate description of the velocity distribution in the boundary layer. The known methods of approximate calculation for a laminar boundary layer are characterized by an assumed set of velocity profiles in the boundary layer, and also by the choice of integral relationships.

Schlichting [103] proposed the first method of approximate calculation for the laminar boundary layer of a porous surface in the presence of suction. This method was based on the integral impulse relationship (1.34). We choose as a set of velocity profiles in the layer the one-parameter set

$$\frac{u}{U} = F_1(\eta) + kF_2(\eta), \tag{3.1}$$

where

$$[illegible]\ \frac{y}{\delta(x)};$$

$$F_1(\eta) = [illegible] - \exp(-\eta);$$

$$F_2(\eta) = F_1 - s[illegible]\eta) \text{ with } 0 < \eta < 3;$$

$$F_2(\eta) = F_1 - 1 \text{ with } \eta > 3.$$

The function $\sin(\frac{\pi}{6}\eta)$ quite satsifactorily approximates the Blasius profile. /78

Due to this choice of the velocity profile, the following boundary conditions are automatically satisfied:

$$\text{with } y=0 \quad u=0;\ v=v_0;\ v_0\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2};$$
$$\text{with } y=\infty \quad u=U;\ \frac{\partial u}{\partial y}=0;\quad \frac{\partial^2 u}{\partial y^2}=0. \tag{3.2}$$

The parameter $k$ is defined by the equation

$$k=\frac{\lambda+\lambda_1-1}{1-\lambda_1\left(1-\frac{\pi}{6}\right)}, \tag{3.3}$$

with

$$\lambda=\frac{dU}{dx}\cdot\frac{\delta^2}{\nu};\quad \lambda_1=\frac{-v_0\delta}{\nu}.$$

This equation was obtained as a result of the substitution of the velocity profiles (3.1) into the impulse relationship (1.34).

To determine the thickness of the displacement flow, the thickness of the impulse loss and the tangential stress distribution, we obtain

$$\left.\begin{aligned}
&\delta^*=\delta(1-0{,}09014k);\\
&\delta^{**}=\delta\frac{1-0{,}09014k}{0{,}5+0{,}06656k-0{,}02358k^2};\\
&\frac{\tau_0\delta^{**}}{\mu U}=\left[1+k\left(1-\frac{\pi}{6}\right)\right](0.5+0.06656k-0{,}02358k^2).
\end{aligned}\right| \tag{3.4}$$

Using equation (1.34) and formula (3.4) we arrive at the ordinary differential equation

$$\frac{1}{2}U\frac{dZ}{dx}+\left[2+\frac{1-k\left(2-\frac{6}{\pi}\right)}{0{,}5+0{,}06656k-0{,}02358k^2}\right]f+t^{**}=\varphi(k), \tag{3.5}$$

where /79

$$Z=\frac{\delta^{**2}}{\nu}=\frac{f}{\frac{dU}{dx}}.\quad t^{**}=\frac{-v_0\delta^{**}}{\nu}.$$

To simplify the notation, we introduce the function

$$G = 2\varphi(k) - 2f\left[2 + \frac{1 - k\left(2 - \frac{6}{\pi}\right)}{0.5 + 0.06656k - 0.02358k^2}\right] - 2t^{**}, \tag{3.6}$$

and reduce equation (3.5) to the form

$$\frac{dZ}{dx} = \frac{1}{U} G. \tag{3.7}$$

It is advisable to integrate equation (3.7) by the isocline method.

The case in which the one-parameter profile set (3.1) undergoes a break in continuity with $\eta = 3$ should be attributed to the deficiencies of the method we are examining. This fact contradicts the physical representations of the boundary layer. In addition, integrating equation (3.7) by the isocline method is a very time-consuming and unwieldy operation.

The Torda method [115] is based on the use of the integral impulse relationships (1.34). For a velocity profile across the boundary layer, we choose a third degree polynomial. The polynomial coefficients $a$, $b$, $c$ and $d$ are defined by the boundary conditions

$$u = 0;\ v = v_0;\ v_0\left(\frac{\partial u}{\partial y}\right)_0 = U\frac{dU}{dx} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right)_0;\quad v_0\left(\frac{\partial^2 u}{\partial y^2}\right)_0 = \nu\left(\frac{\partial^3 u}{\partial y^3}\right)_0$$

$$\text{with } y = 0; \tag{3.8}$$

$$u = U;\quad \frac{\partial u}{\partial y} = 0 \text{ with } y = \delta.$$

The polynomial coefficients are

$$a = \frac{24 + 6\lambda + \lambda t}{\delta D_1};$$

$$b = \frac{3(-3\lambda + 4t)}{\delta^2 D_1};$$

$$c = \frac{-3\lambda t + 4t^2}{\delta^3 D_1};$$

$$d = \frac{-6 + 3\lambda + 2\lambda^2 - 6\lambda t - 3t^2}{\delta^4 D_1}.$$

where $D_1 = 18 + 6t + t^2$, $\lambda = \frac{dU}{dx} \cdot \frac{\delta^2}{\nu}$ the Pohlhausen parameter; $t = \frac{-v_0\delta}{\nu}$ is the suction parameter. /80

Having determined the coefficients of the polynomial, we can compute the function

$$\left.\begin{aligned}
&G_1(\lambda, t)=\frac{\delta^{**}}{\delta}=\frac{1}{1260D_1^2}[144(406+278t+89t^2+13t^3+t^4)-\\
&-6\lambda(132-158t-104t^2-13t^3)-\lambda^2(612+303t-38t^2)];\\
&G_2(\lambda, t)=\frac{\delta^*}{\delta}=\frac{1}{20D_1}[4(36+16t+3t^2)-3\lambda(4+t)];\\
&G_3(\lambda, t)=\frac{24+6\lambda+t\lambda}{D_1}G_1(\lambda, t).
\end{aligned}\right\} \quad (3.9)$$

In this case the impulse relationship has the form

$$G_3\frac{d\delta}{dx}+G_2\left(\frac{U''\delta}{\nu}\right)+G_1\left(\frac{\nu}{U\delta^8}\right)=\frac{d}{dx}\left(\frac{v_0}{\nu}\right). \quad (3.10)$$

Equation (3.10) should also be integrated by the isocline method.

Truckenbrodt [117] proposed a simpler method for calculating a laminar boundary layer on a porous surface in the presence of suction. Converting the impulse equation (1.34) to the form

$$\frac{dz}{dx}=\frac{1}{U}G(f, t^{**}), \quad (3.11)$$

where

$$z=\frac{\delta^{**}}{\nu}; \quad f=\frac{\delta^{**2}}{\nu}\cdot\frac{dU}{dx}; \quad t^{**}=\frac{-v_0\delta^{**}}{\nu},$$

the function $G$ can be represented approximately in linear form /81

$$G=c_0(1-2t^{**})-(b_0+1)f, \quad (3.12)$$

where $c_0$ = 0.441, $b_0$ = 5.

Then equation (3.11) becomes a Bernoulli type equation which can be integrated in finite form. Thus, by computing the function $\delta^{**}(x)$ we can determine all the remaining characteristics of a boundary layer.

To calculate a laminar boundary layer, in the presence of suction, reference [22] used in first and second approximations, a profile set in the form of the sixth degree polynomial

$$\frac{u}{U} = a_1\left(\frac{y}{\delta}\right) + a_2\left(\frac{y}{\delta}\right)^2 + a_3\left(\frac{y}{\delta}\right)^3 + a_4\left(\frac{y}{\delta}\right)^4 + a_5\left(\frac{y}{\delta}\right)^5 + a_6\left(\frac{y}{\delta}\right)^6. \quad (3.13)$$

To determine the coefficients of the polynomial $a_i$ ($i$ = 1, 2, 3, 4, 5, 6), the following basic and supplementary boundary conditions on the surface of the body ($y$ = 0) and on the outer boundary of the layer ($y$ = $\delta$) were used:

$$u = 0;\ v = v_0(x);\ \frac{U}{\nu}\cdot\frac{dU}{dx} + \frac{\partial^2 u}{\partial y^2} = -\frac{v_0}{\nu}\cdot\frac{\partial u}{\partial y};\ \frac{\partial^3 u}{\partial y^3} = -\frac{v_0}{\nu}\cdot\frac{\partial^2 u}{\partial y^2}$$
$$\text{with } y = 0; \quad (3.14)$$
$$u \to U;\ \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial^3 u}{\partial y^3} = 0 \text{ with } y = \delta.$$

The coefficients have the form

$$\left.\begin{aligned}
a_1 &= \frac{(t-12)\lambda - 120}{D_2};\\
a_2 &= \frac{30(\lambda + 2t)}{D_2};\\
a_3 &= \frac{-10t(\lambda + 2t)}{D_2};\\
a_4 &= \frac{5(9t^2 - 36t - 12\lambda + 4\lambda t + 60)}{D_2};\\
a_5 &= \frac{3(20\lambda - 5\lambda t - 12t^2 + 64t - 120)}{D_2};\\
a_6 &= \frac{2(5t^2 - 30t + 2\lambda t - 9\lambda + 60)}{D_2},
\end{aligned}\right\} \quad (3.15)$$

where

$$D_2 = 12t - t^2 - 60.$$

In the particular case of a laminar boundary layer on a non- /82
permeable surface ($t \equiv 0$) the coefficients of (3.15) agree with the respective values computed by Schlichting and Ulrich [101].

Values for $\zeta$, $H$ and $F$ in equation (1.74) as a function of the shape factor of the boundary layer $f$ and the suction parameter $t^{**}$ were computed using polynomial (3.13).[1] Figure 16 shows the results

[1] In references [27,28] a sixth degree polynomial was used to compute the characteristics of a stationary and nonstationary laminar boundary layer on a nonpermeable surface in the presence of blowing.

in graphic form.

It follows from these graphs that the function is almost linear, i.e., as a first approximation we can assume that

$$F(f, t^{**}) = A(t^{**}) - B(t^{**})f. \tag{3.16}$$

Figure 71 shows the curves of the corresponding values of coefficients $A$ and $B$.

According to expression (3.16) we can write the integral of equation (1.74) in the form

$$f(x) = \frac{dU}{dx} \cdot \frac{1}{U^B} \int_0^x [A(x) - 2t^{**}(x)]\, U^{B-1} dx + C \frac{dU}{dx} \cdot \frac{1}{U^B}, \tag{3.17}$$

where $C$ is the integration constant.

When the origin of the coordinate system is placed at the leading critical point ($U = 0$ with $x = 0$), it follows from the condition of finiteness of the parameter $f$ when $x = 0$, that the constant $C = 0$ and $f(0) = \frac{A}{B}$.

The curve $F(f, t^{**})$ (cf. Fig. 16c) does not coincide exactly with a straight line. In the second approximation we can allow for the error arising as a result of the linearization of the function $F(f, t^{**})$. In fact, if we replace expression (3.16) in equation (1.74) with

$$F(f, t^{**}) = A(t^{**}) - B(t^{**})f - \varepsilon(f, t^{**}), \tag{3.18}$$

we obtain the solution to equation (1.74) in the second approximation

$$f(x) = \frac{dU}{dx} U^{-B} \int_0^x [A - 2t^{**} + \varepsilon(f_1 t^{**})]\, U^{B-1} dx. \tag{3.19}$$

The graph of the function $\varepsilon(f, t^{**})$ is shown in Figure 18.

For an approximate calculation of a boundary layer in the presence of suction, Yu.F. Kontsevich and N.D. Shal'kin used a profile set proposed by L.G. Loytsyanskiy [41] for a nonpermeable surface:

$$\frac{u}{U} = 1 + a_1 \left(1 - \frac{y}{\delta}\right)^n + a_2 \left(1 - \frac{y}{\delta}\right)^{n+1} + a_3 \left(1 - \frac{y}{\delta}\right)^{n+2}. \tag{3.20}$$

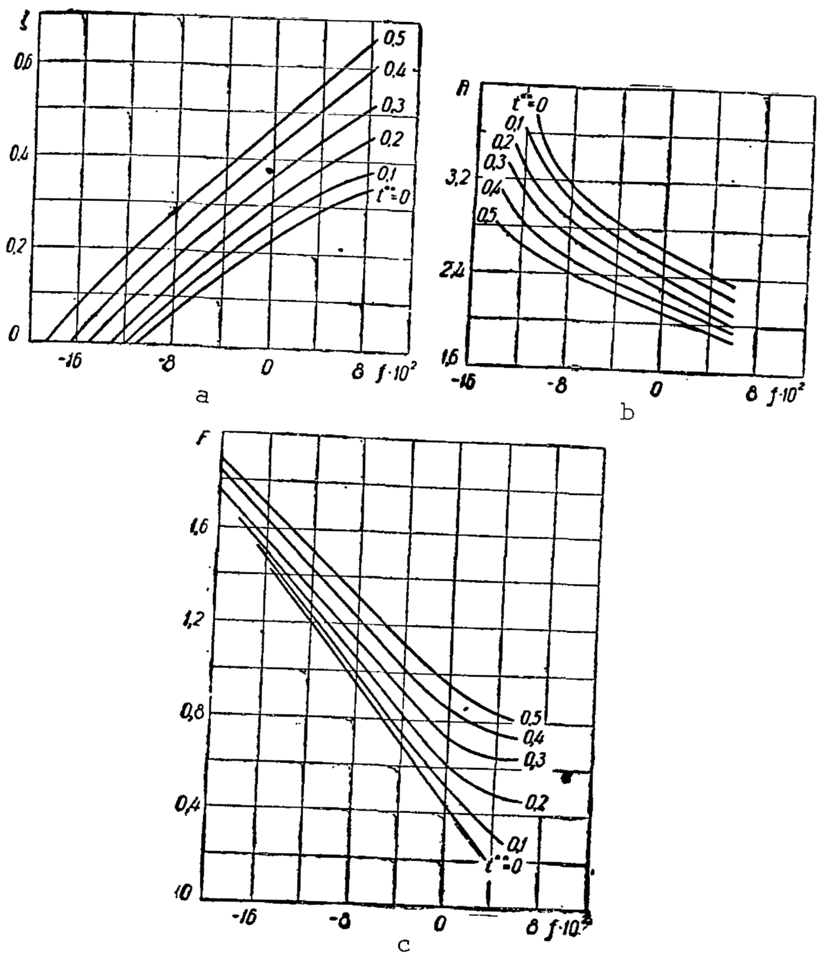
 /83

Fig. 16. The Coefficient of Friction $\zeta$ (a), the Shape Factor $H$ (b) and the Function $F$ (c) as a Function of the Parameters $f$ and $t^{**}$ With an Approximation of the Velocity Distributions Across the Boundary Layer as a Sixth Degree Polynomial.

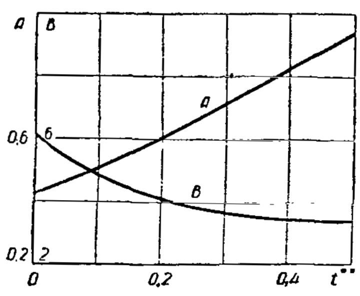


Fig. 17. The Coefficients $A$ and $B$ as a Function of the Suction Parameter With an Approximation of the Velocity Distribution as a Sixth Degree Polynomial.

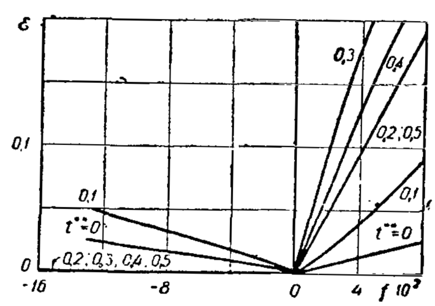
 /84

Fig. 18. The Correction Factor $\varepsilon$ as a Function of the Parameters $f$ and $t^{**}$.

We can determine the coefficients $a_1$, $a_2$, $a_3$, by satisfying the following conditions with $y = 0$:

$$\left.\begin{aligned} & u = 0;\ v = -v_0(x); \\ & \frac{\partial^2 u}{\partial y^2} = -\frac{dU}{dx}\cdot\frac{U}{\nu} - \frac{v_0}{\nu}\cdot\frac{\partial u}{\partial y}; \\ & \frac{\partial^3 u}{\partial y^3} = -\frac{v_0}{\nu}\cdot\frac{\partial^2 u}{\partial y^2}. \end{aligned}\right\} \tag{3.21}$$

Then to compute them we will compile a system of algebraic equations: /85

$$\left.\begin{aligned} & a_1 + a_2 + a_3 = -1; \\ & a_1 n(n-1-f) + a_2(n+1)(n-f) + a_3(n+2)(n+1-f) = -\lambda; \\ & a_1 n(n-1)(n-2-f) + a_2(n+1)n(n-1-f) + \\ & \qquad + a_3(n+2)(n+1)(n-f) = 0. \end{aligned}\right\} \tag{3.22}$$

The index $n$ is chosen from the condition of the best approximation to the velocity profiles with power assignment of the velocities of the external flow and is connected with the parameter $\lambda$ by the function

$$n = 0.15 + \lambda. \tag{3.23}$$

Then for the remaining characteristics of a boundary layer, simple calculations allow us to obtain the following results:

for the thickness of the displacement flow

$$\delta^* = -\delta\left(\frac{a_1}{n+1} + \frac{a_2}{n+2} + \frac{a_3}{n+3}\right); \tag{3.24}$$

for the thickness of the impulse loss

$$\delta^{**} = \delta^* - \delta\left(\frac{a_1^2}{2n+1} + \frac{a_2^2}{2n+3} + \frac{a_3^2}{2n+5} + \frac{a_1 a_2}{n+1} + \frac{2a_1 a_3}{2n+3} + \frac{a_2 a_3}{n+2}\right); \tag{3.25}$$

for the local coefficient of friction

$$\frac{\tau_0}{\varrho U^2} = -\frac{\nu}{U\delta}[na_1 + (n+1)a_2 + (n+2)a_3]. \tag{3.26}$$

Formulas (3.22) - (3.26) allow us to compute the function $F(f, t^{**})$, which we can linearize in first approximation, and then we can integrate the impulse relationship (1.74) in final form.

We should note here that Loytsyankiy's profile set (3.20) was also used by A.P. Girol' to compute the characteristics of a boundary layer in the presence of suction.

## The Simultaneous Use of the Impulse and Energy Relationships

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There are several methods of approximate calculation for a laminar boundary layer in the presence of suction. These methods are based on the simultaneous application of the integral impulse and energy relationships and also on the use of a one-parameter set of velocity profiles. In this sense, the methods we are examining are further developments of the well-known Walz method [118] for laminar boundary layer in the presence of suction.

The method of K. Wieghardt [123] uses the integral relationships of impulses (1.34) and energy (1.37). We choose as velocity profiles the profile set (3.1) proposed by Schlichting. In this case we convert the impulse and energy relationships to the system of equations

$$\begin{aligned} \frac{dH}{d\sigma} &= f(H)\frac{1}{\overline{U}}\cdot\frac{d\overline{U}}{d\sigma} - g(H)\overline{U}\frac{\mathrm{Re}}{R^{**2}} + h(H)\frac{\mathrm{Re}}{R^{**}}\bar{v}_0, \\ \frac{dR^{**}}{d\sigma} &= -\frac{H+1}{\overline{U}}\cdot\frac{d\overline{U}}{d\sigma}R^{**} + \varepsilon\overline{U}\frac{\mathrm{Re}}{R^{**}} + \mathrm{Re}\,\bar{v}_0. \end{aligned} \tag{3.27}$$

Here

$$\sigma = \frac{x}{L};\quad \overline{U} = \frac{U}{U_0};\quad \bar{v}_0 = \frac{v_0}{U_0};\quad \mathrm{Re} = \frac{U_0 L}{\nu};$$

$$f(H) = -\frac{\overline{H}(H-1)}{N};$$

$$g(H) = -\frac{2D - \varepsilon\overline{H}}{N};$$

$$h(H) = -\frac{\overline{H}-1}{N};$$

$$N = \frac{d\overline{H}}{dH},$$

where

$$H = \frac{\delta^*}{\delta^{**}};\quad \varepsilon = \left[\frac{\partial\overline{U}}{\partial\left(\frac{y}{\delta^{**}}\right)}\right]_{y=0};\quad \overline{H} = \frac{\delta^*}{\delta^{***}};$$

$$D = \int_0^\infty y \left[ \frac{\partial \overline{U}}{\partial \left( \frac{y}{\delta^{**}} \right)} \right] d \left( \frac{y}{\delta^{**}} \right).$$ /87

In each concrete case, the system of equations (3.27) should be integrated by the finite difference method.

To compute a laminar boundary layer on a porous surface in the presence of suction, Wuest suggested using the energy equation (1.37) instead of the boundary condition

$$v_0 \left( \frac{\partial u}{\partial y} \right) = U \frac{dU}{dx} + \nu \frac{\partial^2 u}{\partial y^2}. \quad (3.28)$$

The velocity profiles were chosen in the form of (3.1). In this expression we take the new functions

$$F_1(\eta) = 1 - \exp \left[ \frac{\eta}{\sqrt{k}} \right];$$
$$F_2(\eta) = F_1 - \sin \left[ \frac{\pi \eta}{6 \sqrt{k}} \right] \text{with } 0 < \eta < 3. \quad (3.29)$$

As we can see from expression (3.29), the functions $F_1$ and $F_2$ are somewhat different from the functions used by Schlichting. According to Wuest, this change leads to the best results.

The Head method [73] for calculating a laminar boundary layer on a porous surface in the presence of suction, is also based on the integral relationships for impulses (1.34) and energy (1.37). A fourth degree polynomial is used as a velocity profile.

On the outer boundary of the boundary layer the following boundary conditions are satisfied

$$\frac{\partial u}{\partial \eta} = \frac{\partial^2 u}{\partial \eta^2} = 0 \text{ and } \frac{u}{U} = 1 \text{ with } \eta = 1. \quad (3.30)$$

We introduce the notation

$$A = - \left\{ \frac{\partial^2 \left( \frac{u}{U} \right)}{\partial \eta^2} \right\}_{\eta = 0}. \quad (3.31)$$

Then the assigned velocity profile takes the form /88

$$\frac{u}{U} = 2\eta - 2\eta^3 + \eta^4 + \frac{A}{6} \eta (1 - \eta)^3 = F_0(\eta) + A G_0(\eta). \quad (3.32)$$

The curves of the functions $F_0(\eta)$ and $G_0(\eta)$ are shown in Figure 19a and the profile set for negative and positive values of the

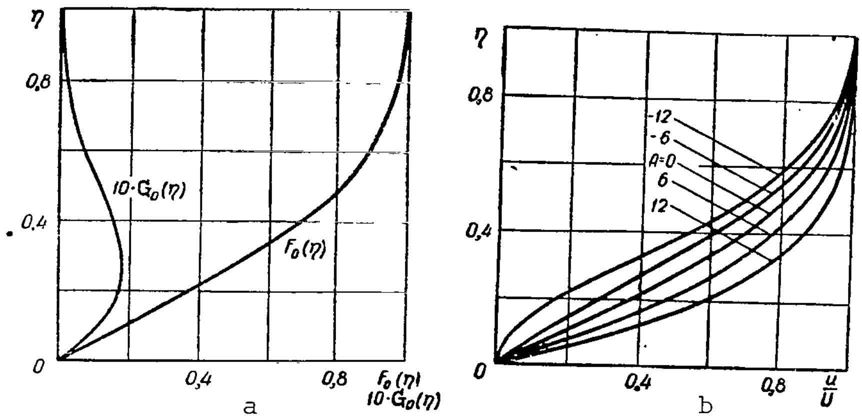


Fig. 19. Graph of the Functions $F_0(\eta)$ and $G_0(\eta)$ (a) and the Set of Velocity Profiles With Suction of a Boundary Layer (b) (According to Head).

shape factor $A$ are shown in Figure 19b. With $A$ = -12 the boundary layer is separated.

The basic parameters in the method under examination are

$$l=\frac{\delta^{**}}{U}\left(\frac{\partial u}{\partial y}\right)_0;\quad m=\frac{\delta^{**2}}{U}\left(\frac{\partial^2 u}{\partial y^2}\right)_0;\quad n=\frac{\delta^{**3}}{U}\left(\frac{\partial^3 u}{\partial y^3}\right)_0;\tag{3.33}$$
$$f=\frac{\delta^{**2}}{\nu}\cdot\frac{dU}{dx};\quad t^{**}=\frac{-v_0\delta^{**}}{\nu}.$$

Using these parameters, we can write the impulse and energy relationships in the form

$$\frac{d}{dx}\left(\frac{\delta^{**2}}{\nu}\right)=\frac{2}{U}\{l-(H+2)f-t^{**}\};\tag{3.34}$$

$$\frac{dH_\varepsilon}{dx}=\frac{1}{U\left(\frac{\delta^{**2}}{\nu}\right)}[2D-H_\varepsilon\{l-(H-1)f-t^{**}\}-t^{**}],\tag{3.35}$$ /89

where $H_\varepsilon=\frac{\delta^{***}}{\delta^{**}}$ is the ratio of conventional thicknesses of the boundary layer; $D=\int_0^\infty\left(\frac{\delta^{**}}{U}\right)^2\left(\frac{\partial u}{\partial y}\right)^2 d\left(\frac{y}{\delta^{**}}\right)$ is the dissipation integral.

In this case we use the boundary conditions

$$f = -m - lt^{**};$$
$$n = -t^{**}m. \tag{3.36}$$

To facilitate calculations in references [73, 74], all the auxiliary functions were tabulated and presented as diagrams convenient for practical use.

## The Use of Integral Equations of Three Moments

In the first chapter in order to compute the three unknown values $H$ $\zeta$ and $f$ we composed a system of three equations which, in essence, are a generalization of the equations for the zero-th, first and second moments, discussed in reference [43], for a porous surface in the presence of suction,

$$a = \frac{1}{H_1 - \frac{1}{2}H_2}; \tag{3.37}$$

$$b = \frac{2H_1 + H_3 + H_4}{H_1 - \frac{1}{2}H_2}; \tag{3.38}$$

$$c = \frac{H_6 + H_7 + \frac{1}{2}H_8}{H_0}; \tag{3.39}$$

$$H_0 = \frac{3H_5 - 2H_6}{4\left(H_1 - \frac{1}{2}H_2\right)}, \tag{3.40}$$ /90

we can write the equations of the first (1.85) and second (1.90) moments in the form

$$\frac{df}{dx} = a\frac{U'}{U}\left(1 - Ht^{**} - \frac{b}{a}f\right) + \frac{U''}{U'}f; \tag{3.41}$$

$$\frac{df}{dx} = \frac{a}{H_0}\cdot\frac{U'}{U}(H - H_4t^{**} - H_0cf) + \frac{U''}{U'}f. \tag{3.42}$$

The range of variation of the shape factor $H$ is comparatively small. For example, for a plate $H$ = 2.0 to 2.59. We assume that in the equation of the first moment (3.41) the shape factor is approximately equal to 2, i.e., to the precise value for an asymptotic boundary layer on a porous plate. Furthermore, using the equation of the second moment to compute the function $H(f,\ t^{**})$, we will subtract equation (3.41) term by term from equation (3.42) and after several simple transformations, we obtain

$$H = H_0 - (2H_0 - H_4)t^{**} - H_0\left(\frac{b}{a} - c\right)f. \qquad (3.43)$$

Assuming in this formula that $t^{**} = 0$ and that formula (3.43) must correspond to the known interpolation formula, discussed in reference [43], we can determine numerical values for the coefficients: $H_0 = 2.59$ and $H_0(\frac{b}{a} - c) = 7.55$.

We can compute the constant $H_4$ according to formula (1.83) using an asymptotic velocity profile in the boundary layer on a porous plate (2.3). As a result of these calculations we find that $H_4 = 4$.

Thus we can determine numerical values for the constant coefficients and write the interpolation formula in its final form:

$$H = 2{,}59 - 1{,}18t^{**} - 7{,}55f. \qquad (3.44)$$

A comparison of this formula with the data from the integration of the equations for a laminar boundary layer on a porous plate shows a satisfactory correlation (Fig. 20). Comparison of analogous data for a porous wedge [98] showed that with $t^{**} = 0.462$ and $f = 0.01675$, the precise value of $H$ was $H = 2.03$ and the approximate value was $H = 1.97$.

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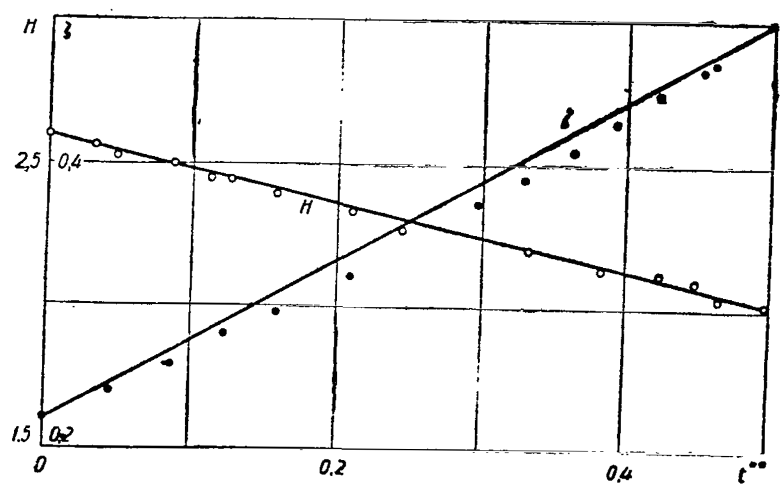


Fig. 20. Comparison of the Results of Calculations According to the Approximate Formulas (3.44) and (3.46) (Solid Lines) With the Precise Solutions [62] (Dots).

The function thus obtained, $H(f, t^{**})$(Fig. 21), was compared with the known approximate solutions. This graph allows us to evaluate the degree of convergence of the obtained analytic formula and the known approximate graphic functions.

Simultaneously solving the equations of the zeroth and first moments, we obtain

$$\zeta=\frac{a}{2}+(1-a)t^{**}+\left(2+H+\frac{b}{2}\right)f. \tag{3.45}$$

We assume that $t^{**}$ = 0; then with the correlation of equation (3.45) to the known solution [43], we can determine numerical values for the coefficients: $a$ = 0.4408 and $b$ = 5.48.

Using formula (3.44) and the values of the coefficients $a$ and $b$ from equation (3.45) we obtain

$$\zeta=0.22+0.56t^{**}-1.18ft^{**}+1.85f-7.55f^2. \tag{3.46}$$

Comparison of the data computed by this formula with the precise values of $\zeta$ for a porous plate [62] shows a satisfactory correlation in the interval of parameter values 0 - 0.5 (cf. Fig. 20). The maximum error in this case does not exceed 5%.

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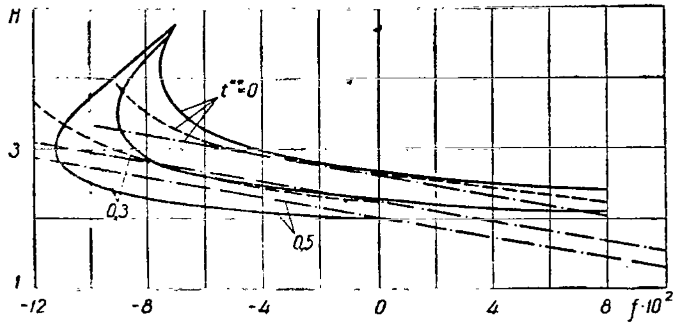


Fig. 21. Comparison of the Values of $H$, Computed by Formula (3.44), with the Data of Different Authors: ——— Shows the Data of [103]; ····· Shows the Data of Yu.F. Kontsevich and N.D. Shal'kin; -·- Shows the data According to Formula (3.44).

It is interesting to compare the precise numerical data, computed in reference [98] for a porous wedge, with the results of the calculations according to the formulas we obtained. The comparision showed that for $t^{**}$ = 0.462 and $f$ = 0.01675, the precise value of $\zeta$ was 0.488 and the approximate solution, computed by formula (3.46), was 0.498. A comparison of the approximate functions obtained earlier, with the results of the calculations according to formula (3.46), is made in Figure 22.

Placing the value $\zeta$ = 0 into equation (3.46), we obtain a formula for the value of the shape factor $f_s(t^{**})$ at the separation point of the boundary layer:

$$f_s = \frac{(1.85 - 1.18t^{**}) - \sqrt{(1.85 - 1.18t^{**})^2 + 4 \cdot 7.55(0.22 + 0.56t^{**})}}{2 \cdot 7.55} \quad (3.47)$$

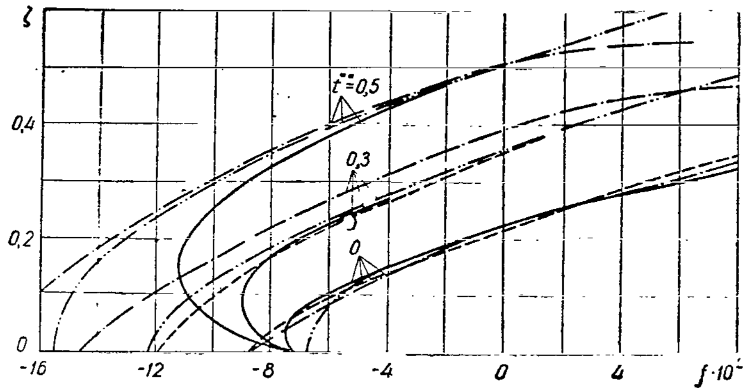


Fig. 22. Comparison of the Values of ζ, Computed by Formula (3.46) with the Data of Different Authors: —— Shows the Data of [103]; ··· Shows the Data of Yu.F. Kontsevich and N.D. Shal'kin; -·- Shows the Data According to Formula (3.46) and -··- Shows the Data of [117].

For a nonpermeable surface according to formula (3.47), $f_s = -0.0875$.

Figure 23 shows a comparison of the known data, included in formula (3.47), according to the influence parameter $t^{**}$ on the value of the shape factor at the separation point of the boundary layer. /93

To calculate the shape factor $f$ we can use any of three equations of moments, since, as the calculations obtained in this case show, the results are very similar. In subsequent determining calculations, we will use the equation of the zeroth moment in the following form:

$$\frac{df}{dx} = \frac{U''}{U'} f + \frac{U'}{U}(F - 2t^{**}). \quad (3.48)$$

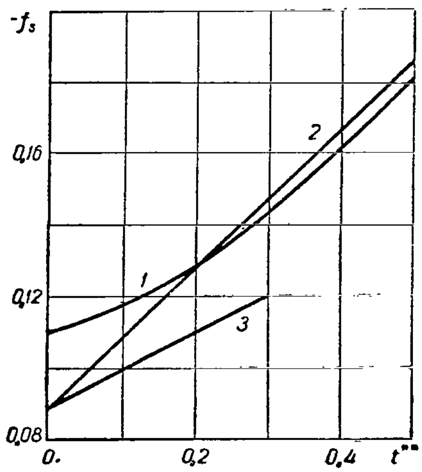


Fig. 23. The Values of the Shape Factor at the Separation Point of the Layer $f_s$ as a Function of the Suction Parameter $t^{**}$: "1" is Given According to the Data of [22]; "2" According to the Data of [13] and "3" According to the Data of Yu.F. Kontsevich and N. D. Shal'nik.

Here

$$F(f, t^{**}) = 2\zeta(f, t^{**}) - \\ - 2[2 + H(f, t^{**})] f. \tag{3.49}$$

Substituting values of $H$ and $\zeta$ into formula (3.49), we can demonstrate that the function $F$ is a linear function of the shape factor $f$, i.e.

$$F(f, t^{**}) = A(t^{**}) + B(t^{**}) f, \\ A(t^{**}) = 0.44 + 1{,}12t^{**}, \quad B = 5.48. \tag{3.50}$$

Comparison of the obtained function $F(f, t^{**})$ with the earlier known approximate graphic functions (Fig. 24) allows us to evaluate their concurrence. /94

Since $F(f, t^{**})$ is a linear function, then the integral of eqation (3.48) can be written in the form

$$f(x) = \frac{U'(x)}{[U(x)]^B} \int_0^x [A(x) - 2t^{**}(x)] [U(x)]^{B-1} dx + C \frac{U'(x)}{U^B(x)}. \tag{3.51}$$

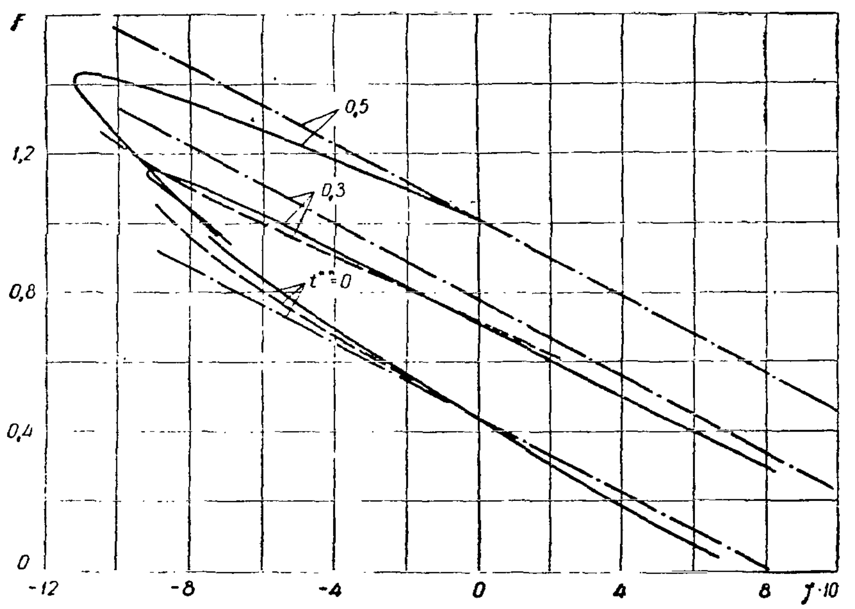


Fig. 24. The Function $F(f, t^{**})$: —— is According to the Data of [103]; ···· is According to the Data of Yu.F. Kontsevich and N.D. Shal'kin; -·- is According to Formula (3.50).

We place the coordinate system in such a way that when $x = 0$, $U = 0$. Then from the condition of the finiteness of $f$, when $x = 0$ the integration constant $C = 0$, and $f(0) = \frac{A}{B}$. Figure 25 shows the shape factor at the leading critical point as a function of the suction parameter.

Computing the values of the shape factor $f(x)$ according to formula (3.15), we can determine the remaining characteristics of a laminar boundary layer: /95

$$\left.\begin{aligned} \delta^{**} &= \sqrt{\frac{\nu f(x)}{U(x)}};\\ \delta^{*} &= \delta^{**} H(f, t^{**});\\ \frac{\tau_0}{\frac{1}{2}\varrho U^2} &= \frac{2\nu}{U\delta^{**}}\zeta(f, t^{**});\\ v_0 &= \frac{\nu \dot{t}^{**}}{\delta^{**}}. \end{aligned}\right| \qquad (3.52)$$

In order to calculate the characteristics of a boundary layer with a prescribed normal component of velocity on the porous surface of a body $v_0$, we must use the method of successive approximations.

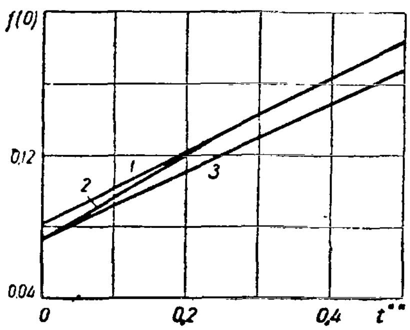


Fig. 25. The Values of the Shape Factor $f(0)$ at the Critical Point as a Function of the Suction Parameter $t^{**}$: "1" Shows the Function According to the Data of [13]; "2" According to the Data of Yu. F. Kontsevich and "3" According to the Data of [117].

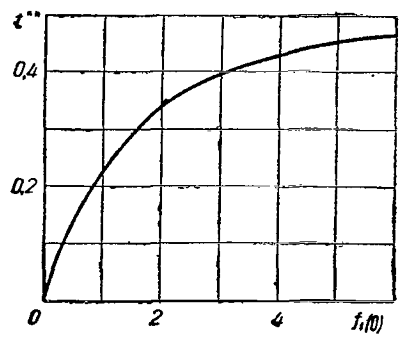


Fig. 26. The Function $f_1(0)$ as a Function of the Suction Parameter $t^{**}$.

Taking in the first approximation the expected change in the parameter $t^{**}$, we can calculate by formulas (3.51) and (3.52) the function $\delta^{**}$ and by the values of this function, since $v_0$ is given, find the change in the parameter $t^{**}$ in the second approximation. Repeating the process of successive approximations, we can calculate with sufficient accuracy all the characteristics of a boundary layer.

As a first approximation of the expected change in the parameter $t^{**}$ we should take the corresponding values for a porous /96
plate with a given distribution $v_0$ for a body with a porous surface. The necessary values of $t^{**}$ for a porous plate should be determined in the following way. First we should compute the values of $f_1(0)$ according to the formula

$$f_1(0) = \frac{\frac{v_0}{U}}{2\sqrt{\frac{Ux}{\nu}}}, \tag{3.53}$$

and then determine the expected change in the parameter $t^{**}$ in first approximation according to the graph (Fig. 26) [17]. After computing the values of the shape factors $f$ and $t^{**}$ with the necessary accuracy, we can determine the remaining characteristics of a boundary layer according to formulas (3.52).

The method under discussion can also be used to calculate the characteristics of a laminar boundary layer and the function $v_0(x)$ according to the prescribed values of the functions $\delta^{**}$, $U$ and $\frac{dU}{dx}$. If the enumerated values are prescribed, we can calculate the functions $f$ and $\frac{df}{dx}$ and from equations (3.48) and (3.52) determine the unknown functions $t^{**}$ and $v_0$.

In practice, the case is interesting when the suction rate is roughly proportional to the thickness of the impulse loss of the boundary layer, i.e., $y_0$ $\delta^{**}$. In this case, $t^{**} = t_0^{**} = \text{const}$, and equation (3.51) is simplified and converted to the form

$$f(x) = \frac{0{,}44\,[1 - 2t_0^{**}]}{U^B(x)} \cdot \frac{dU}{dx} \int_0^x U^{B-1} dx. \tag{3.54}$$

We will also examine the particular case of uniform suction of a laminar boundary layer on a plate [20]. In this case $v_0$ = const. and the equation of the zero-th moment can be converted to the form

$$\frac{d\delta^{**}}{dx} + \frac{v_0}{U} = \frac{\tau_0}{\varrho U^2}. \tag{3.55}$$

Using equation (3.55) we determine the characteristics of a laminar boundary layer with uniform suction along the entire surface of the plate. We now introduce the new variables /97

$$\xi = \left(\frac{-v_0}{U}\right)^2 \frac{Ux}{\nu}; \qquad t^{**} = \frac{-v_0 \delta^{**}}{\nu}$$

and transform equation (3.55)

$$t^{**} \frac{dt^{**}}{d\xi} - t^{**} = \zeta. \tag{3.56}$$

We know that with an arbitrary distribution of the suction rate along a porous plate, the coefficient of local friction is

$$\zeta = \zeta_0 - dt^{**}, \tag{3.57}$$

where $\zeta_0 = 0.22$; $d = 0.56$ and $t^{**} < 0$. A comparison of the results of the calculations according to formula (3.57) with the results of numerical integration of the equations for a laminar boundary layer on a calculating machine showed that in the interval of $t^{**}$ values from 0 to -0.5, the maximum error does not exceed about 3%.

Let us now transform equation (3.56) taking formula (3.57) into account. After separating the variables and determining the integration limits, we obtain the simple integral equation

$$\int_0^{t^{**}} \frac{t^{**} dt^{**}}{\zeta_0 + (1-d) t^{**}} = \int_0^{\xi} d\xi \tag{3.58}$$

with the boundary conditions

$$t^{**} = t^{**} \text{ with } \xi = \zeta;$$
$$t^{**} = 0 \text{ with } \xi = 0.$$

Equation (3.58) is solved in the quadratures

$$\xi = \frac{t^{**}}{1-d} - \frac{\zeta_0}{(1-d)^2} \ln\left|1 + \frac{(1-d)}{\zeta_0} t^{**}\right|. \tag{3.59}$$

In reference [78], a precise solution was found to the problem of determining the characteristics of an incompressible laminar boundary layer with uniform suction on a porous plate by numerical integration of the Prandtl differential equations. Figure 27 compares the results of the precise solution (the dots) with the /98 approximate values, computed according to formula (3.59) (the line).

## Use of the Targ and Shvets Approximate Methods

Approximate methods which are simple and very similar in idea, for calculating a laminar boundary layer for a non-porous surface, were developed by M.Ye. Shvets [54] and S.M. Targ [51]. In reference [53] these methods were applied to a laminar boundary layer in the presence of suction.

Let us examine both these methods as applied to a laminar boundary layer in the presence of suction. Using the equation of continuity (1.28), we exclude from the basic Prandtl differential equation (1.27) the transverse velocity component and reduce the equation to the form

$$\nu\frac{\partial^2 u}{\partial y^2}+U\frac{dU}{dx}=u\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\left(v_0-\int_0^y\frac{\partial u}{\partial x}dy\right). \quad (3.60)$$

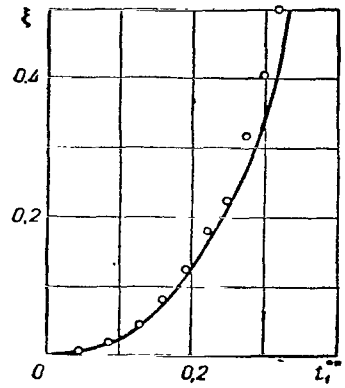


Fig. 27. Comparison of the Precise and Approximate Solutions for Uniform Suction of a Laminar Boundary Layer on a Plate.

The right-hand side of this equation is replaced by its approximate value, which was obtained as a result of replacing the velocity with its approximate value in the form of a polynomial. Afterwards, both sides of the obtained equation are integrated twice over $y$ with satisfaction of the usual boundary conditions.

Targ proceeded from an initial velocity profile in the form of the third degree polynomial

$$\frac{u}{U}=\frac{1}{2}(3\eta-\eta^3), \quad (3.61)$$

where $\eta = \frac{y}{\delta}$.

Substituting the velocity profile (3.61) into the right-hand side of equation (3.61), we obtain /99

$$\nu\frac{\partial^2 u}{\partial y^2}+U\frac{dU}{dx}=\frac{U}{16}\cdot\frac{dU}{dx}(18\eta^2-3\eta^4+\eta^6)-\frac{3}{16}\cdot\frac{U^2}{\delta}\cdot\frac{d\delta}{dx}(6\eta^2-7\eta^4+\eta^6)+\frac{3}{2}\cdot\frac{v_0U}{\delta}(1-\eta^2). \quad (3.62)$$

Equation (3.62) was obtained as a result of an approximate substitution of the acceleration profile in a certain cross section of the boundary layer in the presence of longitudinal pressure drop, by its value for an equivalent plate.

After double integration over $y$ of the differential equation (3.62) with satisfaction of the adhesio- condition and the boundary conditions (1.19), we obtain in second approximation the velocity distribution across a laminar boundary layer in the presence of suction:

$$\frac{u}{U}=\frac{1}{16}\cdot\frac{dU}{dx}\cdot\frac{\delta^2}{\nu}\left(\frac{366}{35}\eta-8\eta^2+\frac{3}{2}\eta^4-\frac{1}{10}\eta^6+\frac{1}{56}\eta^8\right)+$$
$$+\frac{3}{16}\cdot\frac{U\delta}{\nu}\cdot\frac{d\delta}{dx}\left(\frac{26}{35}\eta-\frac{1}{2}\eta^4+\frac{7}{30}\eta^6-\frac{1}{56}\eta^8\right)+ \qquad (3.63)$$
$$+\frac{\delta v_0}{\nu}\left(\frac{3}{4}\eta^2-\frac{1}{8}\eta^4-\eta\right).$$

The process of finding a third approximation is very unwieldy and would not significantly improve the precision of the solution.

It remains for us to find an ordinary differential equation to determine the only unknown parameter, the thickness of the boundary layer. According to S.M. Targ, to do this requires for the velocity profile (3.63) satisfaction of the following condition on the outer boundary of the layer: $u = U$ when $y = \delta$. As a result we obtain the ordinary differential equation

$$\frac{Ud\delta^2}{\nu dx}+5{,}64\frac{dU}{dx}\cdot\frac{\delta^2}{\nu}-\frac{3}{8}v_0\frac{\delta}{\nu}=23{,}27. \qquad (3.64)$$

We integrate equation (3.64) by the method of successive approximations. In the first approximation we examine the boundary layer on a nonpermeable surface (i.e., $v_0 \equiv 0$). In this case equation /100
(3.64) is integrated in the quadratures:

$$\frac{\delta^2}{\nu}=\frac{23{,}27}{[U(x)]^{5{,}64}}\int_0^x U^{4{,}64}(\xi)\,d\xi. \qquad (3.65)$$

M.Ye. Shvets[2] computes the zero-th approximation, by substituting $u = 0$ into the right-hand side of equation (3.60), and then by repeated double integration over $y$ obtains an expression for the velocity profile. For a laminar boundary layer in the presence of

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[2] A certain modification of this method was discussed in reference [4].

suction, analogous computations in the second approximation allow us to obtain

$$\frac{u}{U}=\frac{1}{24}\cdot\frac{dU}{dx}\cdot\frac{\delta^2}{\nu}(\eta^4-12\eta^2+11\eta)-\frac{1}{24}\frac{U\delta}{\nu}\cdot\frac{d\delta}{dx}(\eta^4-\eta)+$$
$$+1\eta+\frac{\delta v_0}{2\nu}(\eta^2-\eta). \qquad (3.66)$$

Satisfying the boundary condition $\frac{\partial u}{\partial y}=0$ when $y=\delta$, we obtain the ordinary differential equation for determining the thickness of the boundary layer:

$$\frac{3}{8}\cdot\frac{dU}{dx}\cdot\frac{\delta^2}{\nu}+\frac{1}{8}\cdot\frac{U\delta}{\nu}\cdot\frac{d\delta}{dx}-\frac{1}{2}v_0\frac{\delta}{\nu}=1. \qquad (3.67)$$

The equation thus obtained differs from the analogous equation (3.64) only in terms of coefficients.

Using Newton's law and expressions (3.63) and (3.66), we obtain a corresponding relationship for determining the coefficient of local friction:

$$\frac{\tau_0}{\varrho U^2}=0.261\frac{\nu}{U\delta}\left(\frac{dU}{dx}\cdot\frac{\delta^2}{\nu}+6.21-1.51\frac{v_0\delta}{\nu}\right); \qquad (3.68)$$

$$\frac{\tau_0}{\varrho U^2}=\frac{1}{3}\cdot\frac{\nu}{U\delta}\left(\frac{dU}{dx}\cdot\frac{\delta^2}{\nu}+4-\frac{v_0\delta}{\nu}\right). \qquad (3.69)$$

To determine the separation point ($\tau_0=0$) from expression (3.68) and (3.69), we obtain

$$\frac{dU}{dx}\cdot\frac{\delta^2}{\nu}+6.21-1.51\frac{v_0\delta}{\nu}=0; \qquad (3.70)$$ /101

$$\frac{dU}{dx}\cdot\frac{\delta^2}{\nu}+4-\frac{v_0\delta}{\nu}=0. \qquad (3.71)$$

We will now examine in more detail the particular case of a plate with $\frac{dU}{dx}=0$) [3]. Taking the expression $u=U\frac{y}{\delta}$ as a first approximation, we obtain in the second approximation

$$\frac{u}{U}=\frac{y}{\delta}+\frac{1}{24}\cdot\frac{U^2}{\nu}\delta\left(y-\frac{y^4}{\delta^3}\right)+\frac{Uv_0}{2\nu}\left(\frac{y^2}{\delta}-y\right). \qquad (3.72)$$

We can find the thickness of the boundary layer $\delta(x)$ from the additional boundary condition $\frac{\partial u}{\partial y} = 0$ when $y = \delta$. From this, taking $\delta(0) = 0$ into account, we obtain by successive approximations

$$\frac{d\delta}{dx} = \frac{8\nu}{U\delta} + 4\frac{v_0}{U}; \tag{3.73}$$

$$\delta - \frac{2\nu}{v_0}\ln\left(1 + \frac{v_0}{2\nu}\delta\right) = 4\frac{v_0}{U}x. \tag{3.74}$$

Further on we will examine the case $\frac{v_0\delta}{\nu} \ll 1$. In formula (3.74) we expand the logarithm into a series and, limiting ourselves to the first three terms, we obtain

$$\delta^2 - \frac{v_0}{3\nu}\delta^3 = 16\frac{\nu x}{U}; \tag{3.75}$$

$$\delta = 4\sqrt{\frac{\nu x}{U}} + \frac{8}{3}\cdot\frac{v_0}{U}x. \tag{3.76}$$

It follows from formula (3.76) that for an asymptotic boundary layer

$$\delta_{as} = \frac{\nu}{v_0}.$$

From these expressions we can obtain approximate asymptotic functions for the velocity profile and the friction stress

$$\frac{u_{as}}{U} = \frac{v_0 y}{\nu} - \frac{1}{4}\left(\frac{v_0 y}{\nu}\right)^2; \tag{3.77}$$ /102

$$\tau_{as} = \varrho U v_0. \tag{3.78}$$

Comparisons showed that expression (3.77) was very close to the known precise solution (2.3), and the asymptotic value of the friction stress on the surface of the plate (3.78) completely agrees with the precise value.

## A Nonstationary Boundary Layer

In the coordinate system associated with a moving body in an incompressible fluid without rotation, the basic differential equation of motion and the equation of continuity for a nonstationary laminar boundary layer have the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}; \tag{3.79}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.$$

On a porous surface of a body in the presence of suction, the following boundary and initial conditi-ns must be satisfied:

$$\begin{aligned} u=0,\quad v=-v_0 &\text{ with } y=0; \\ u\to U(x,t) &\text{ with } y\to\infty; \\ u=U(x,y) &\text{ with } t=0. \end{aligned} \tag{3.80}$$

The first equation of system (3.79) with the help of the second, can be converted to a form which will be more convenient for further calculations:

$$\frac{\partial(U-u)}{\partial t}+\frac{\partial}{\partial x}[u(U-u)]+\frac{\partial U}{\partial x}(U-u)-\nu\frac{\partial^2(U-u)}{\partial y^2}=0. \tag{3.81}$$

Multiplying both sides of equation (3.81) by $y^k$, where $k$ = 1, 2, 3, ..., we will integrate it over $y$ from zero to infinity. Since

$$\int_0^\infty y^k-\frac{\partial(U-u)}{\partial t}\,dy=\frac{\partial}{\partial t}\int_0^\infty y^k(U-u)\,dy, \tag{3.82}$$

after simple transformations we obtain /103

$$\frac{\partial}{\partial t}\int_0^\infty y^k(U-u)\,dy+\frac{\partial}{\partial x}\int_0^\infty y^k u(U-u)\,dy+\int_0^\infty y^k\frac{\partial}{\partial y}[U(U-u)]\,dy+$$
$$+\frac{\partial U}{\partial x}\int_0^\infty y^2(U-u)\,dy-\nu\int_0^\infty y^k\frac{\partial^2(U-u)}{\partial y^2}\,dy=0. \tag{3.83}$$

We will now examine the first term of equation (3.83), assigning to the parameter $k$ values equal to 0, 1, and 2 respectively.

with $k=0$

$$\frac{\partial}{\partial t}\int_0^\infty(U-u)\,dy=\frac{\partial}{\partial t}U\int_0^\infty\left(1-\frac{u}{U}\right)dy=\frac{\partial(U\delta^*)}{\partial t}; \tag{3.84}$$

with $k=1$

$$\frac{\partial}{\partial t}\int_0^\infty y(U-u)\,dy=\frac{\partial(H_4U\delta^{**2})}{\partial t}; \tag{3.85}$$

with $k=2$

$$\frac{\partial}{\partial t}\int_0^\infty y^2(U-u)\,dy = \frac{\partial(H_8 U\delta^{**3})}{\partial t} \tag{3.86}$$

The remaining terms of equation (3.83) can be transformed analogously to the way it was done for a stationary boundary layer.

According to V.V. Struminskiy [49], for a nonstationary flow we can generalize the equations for a laminar boundary layer on a porous surface in the presence of suction. We assume that as velocity profile sets in the cross sections of a nonstationary layer we can use profiles determined by the expression

$$\frac{u}{U}=\varphi\left(\frac{y}{\delta^{**}},\,t\right), \tag{3.87}$$

with the shape factor $f$ and the value $\delta^{**}$ as functions of the coordinate $x$ and the time $t$.

This assumption corresponds to a quasistationary method for examining phenomena in a nonstationary boundary layer. We note that for a nonstationary boundary layer the shape factor /104

$$f=\frac{\delta^{**2}}{\nu}\left(\frac{1}{U}\cdot\frac{\partial U}{\partial t}+\frac{\partial U}{\partial x}\right). \tag{3.88}$$

For the particular case $k = 0$, equation (3.83) is converted to the form

$$\begin{gathered}\Phi(f,t)\frac{\partial f}{\partial t}+U\frac{\partial f}{\partial x}+[N(x,t)H(f,t)+M(x,t)]f=\\=\left(\frac{\partial U}{\partial x}+\frac{1}{U}\cdot\frac{\partial U}{\partial t}\right)[F(f,t^{**})-2t^{**}].\end{gathered} \tag{3.89}$$

Here

$$\left.\begin{aligned}&\Phi(f,t)=H(f,t^{**})+2f\frac{dH}{df};\\&M(x,t)=\frac{\dfrac{\partial U}{\partial t}\cdot\dfrac{\partial U}{\partial x}+U\dfrac{\partial^2 U}{\partial x^2}+U\dfrac{\partial^2 U}{\partial x\,\partial t}}{U\dfrac{\partial U}{\partial x}+\dfrac{\partial U}{\partial t}}-4\frac{1}{U}\cdot\frac{\partial U}{\partial t},\end{aligned}\right| \tag{3.90}$$

$$N(x,t)=\frac{\left(\frac{\partial U}{\partial t}\right)^2-U\frac{\partial^2 U}{\partial t^2}-U^2\frac{\partial^2 U}{\partial x\partial t}}{U\left(U\frac{\partial U}{\partial x}+\frac{\partial U}{\partial t}\right)};$$

$$F(f,t^{**})=\{2\zeta(f,t^{**})-2[2-H(f,t^{**})]\}f;$$

$$\zeta=\frac{\tau_0\delta^{**}}{\mu U},\qquad t^{**}=\frac{v_0\delta^{**}}{\nu}.$$

For a nonpermeable surface, equation (3.89) becomes the equation of V.V. Struminskiy [49]. Due to the assumption of the quasi-stationarity, the values of the functions $\zeta$, $H$ and $F$ have the same form, as in the case of a stationary boundary layer, since they are time functions only through the shape factor $f$. In connection with this fact, we can use interpolation formulas for $\zeta$, $H$ and $F$ in our calculations. Using these formulas, we obtain

$$\Phi(f,t^{**})=H_0-(2H_0-H_4)t^{**}-3H_0\left(\frac{b}{a}-c\right), \qquad (3.91)$$ /105

and convert equation (3.89) to the form

$$\left[H_0-(2H_0-H_4)t^{**}-3H_0\left(\frac{b}{a}-c\right)f\right]\frac{\partial f}{\partial t}+U\frac{\partial f}{\partial x}+$$
$$+N(x,t)\left[H_0-(2H_0-H_4)t^{**}-H_0\left(b-\frac{a}{c}\right)f\right]+M(x,t^{**})f= \qquad (3.92)$$
$$=\left(\frac{\partial U}{\partial x}+\frac{1}{U}\cdot\frac{\partial U}{\partial t}\right)\{[A(t^{**})-B(t^{**})f]-2t^{**}\},$$

where $M(x,t)$ and $N(x,t)$ are prescribed functions.

This method for computing the characteristics of a nonstationary laminar boundary layer can be used, if $f \leq 0.06$. However, it is not sufficiently precise for the flow in the region of separation of the boundary layer. In this case we must obtain more precise values for the functions $\Phi$, $F$, $H$ and $\zeta$. To do this we can use the results of the precise calculations by Terril [110].

Later we will examine the integral impulse relationship (3.83), which for the case $k = 0$ we can write in the form

$$\frac{1}{U}\cdot\frac{\partial\delta^*}{\partial t}+\frac{1}{U^2}\cdot\frac{\partial U}{\partial t}\delta^*+\frac{\partial\delta^{**}}{\partial x}+\frac{1}{U}\cdot\frac{\partial U}{\partial x}(2\delta^*+\delta^{**})+v_0=$$
$$=\frac{\nu}{U^2}\left(\frac{\partial u}{\partial y}\right)_{y=0}=\frac{\tau_0}{\varrho U^2}. \qquad (3.93)$$

In this relationship the values of $u$, $v$, $U$, $\delta^*$, $\delta^{**}$ and $\tau_0$ are functions in the two variables $x$ and $t$.

According to L.A. Rozin [44], as the conventional thickness of the boundary layer in expression (3.93) we take the thickness of the displacement loss $\delta^*$. This allows us to linearize with the best approximation relationship (3.93) when we integrate it. We now convert the integral relationship (3.93) to the form

$$\frac{\partial f^*}{\partial t} + U\left(K + 2f^*\frac{\partial K}{\partial f^*}\right)\frac{\partial f^*}{\partial x} = \left[\frac{\partial \Omega}{\partial t}\cdot\frac{1}{\Omega} - 2\Omega + U\left(\frac{\partial \Omega}{\partial x}\cdot\frac{1}{\Omega} - 4\frac{\partial U}{\partial x}\cdot\frac{1}{U}\right)K - 2\frac{\partial K}{\partial t^*}\cdot\frac{\partial t^*}{\partial x}\right]f^* + 2\Omega(t^* + \zeta^*). \qquad (3.94)$$

Here

$$f^* = \frac{\Omega\delta^{*2}}{\nu};\qquad \zeta^* = \left[\frac{\partial(u/U)}{\partial(y/\delta^*)}\right]_{y=0};\qquad K = \frac{\delta^{**}}{\delta^*} = \frac{1}{H};$$
$$t^* = \frac{v_0\delta^*}{\nu};\qquad \Omega = \frac{\partial U}{\partial x} + \frac{1}{U}\cdot\frac{\partial U}{\partial t}. \qquad (3.95)$$ /106

Using the hypothesis of quasistationarity, from the equation

$$f^* = fH^2;\quad \zeta^* = \zeta H;\quad K = \frac{1}{H},\quad t^* = t^{**}H;$$
$$f = \frac{\Omega\delta^{**2}}{\nu};\quad \zeta = \left[\frac{\partial(u/U)}{\partial(y/\delta^{**})}\right]_{y=0};\quad H = \frac{\delta^*}{\delta^{**}};\quad t^{**} = \frac{v_0\delta^{**}}{\nu} \qquad (3.96)$$

we can calculate the necessary values for a nonstationary boundary layer.

The values of $f$, $\zeta$, $H$ and $t^{**}$ in equations (3.96) were calculated for different values of the suction parameter $t^{**}$, assigning a shape to the velocity profile in the cross section of the boundary layer in the form of a sixth degree polynomial (3.13).

Analysis of the obtained functions $\zeta^*(f^*, t^*)$ and $K(f^*, t^*)$ (Fig. 28) showed that these functions can be linearized with reasonable accuracy, i.e.,

$$K = \alpha_1(t^*), \qquad (3.97)$$
$$\zeta^* = \alpha_2(t^*) + \alpha_3(t^*)f^*. \qquad (3.98)$$

The function $\alpha_1(t^*) = K_S(t^*)$ is shown in Figure 28b by the broken line. Figure 29 shows the curves of the functions $\alpha_2(t^*)$

and $\alpha_3(t^*)$. The straight lines (3.98), approximating the function $\zeta^*(f^*, t^*)$, were drawn through the separation point of the boundary layer and the values of $\zeta^*$ for the case of a plate ($f^* = 0$).

Taking expressions (3.97) and (3.98) into account, we transform the integral relationship (3.94) to the form

$$f^* + \alpha_1 U\frac{\partial f^*}{\partial x} - \left[\frac{\partial\Omega}{\partial t}\cdot\frac{1}{\Omega} - 2\Omega + \alpha_1 U\left(\frac{\partial\Omega}{\partial x}\cdot\frac{1}{\Omega} - 4\frac{\partial U}{\partial x}\cdot\frac{1}{U}\right) + 2\alpha_3\Omega - 2\frac{\partial\alpha_1}{\partial t^*}\cdot\frac{\partial t^*}{\partial x}\right] f^* = 2\Omega(\alpha_2 + t^*). \tag{3.99}$$

For a prescribed velocity distribution on the outer boundary of the layer, equation (3.99) can be integrated by known methods. The corresponding system of characteristics is easy to determine. /107

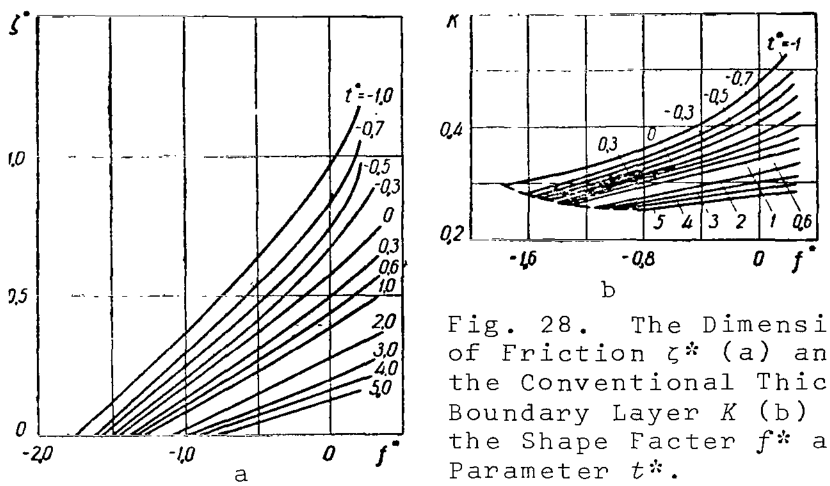



Fig. 28. The Dimensionless Coefficient of Friction $\zeta^*$ (a) and the Ratios of the Conventional Thicknesses of the Boundary Layer $K$ (b) as a Function of the Shape Facter $f^*$ and the Suction Parameter $t^*$.

In a number of cases the solution can be reduced to quadratures, but in the more complex cases approximate methods must be used.

We can compute the value of the parameter $f_s^*$ at the separation point of the boundary layer ($\zeta^* = 0$) from expression (3.98)

$$f_s^* = -\frac{\alpha_2}{\alpha_3}. \tag{3.100}$$

Figure 30 shows the curve of the function $f_s^*(t^*)$.

After computing $f^*$ and consequently $\delta^*$ for each value of $t^*$, we can determine all the remaining characteristics of a nonstationary laminar boundary layer in the presence of suction.

We will now examine the problem of the formation of a nonstationary boundary layer in the presence of suction or blowing on

a stationary circular cylinder, which is suddenly set in uniform motion. We will determine a distance $\sigma$ or a time interval $t_s$ between the beginning of the cylinder's motion and the moment when an eddy in the trailing end causes separation. In the particular /108
case under examination the velocity distribution on the outer boundary of the boundary layer [44] is

$$U(x, t) = 2U_0 \sin\left(\frac{x}{a}\right). \qquad (3.101)$$

As a result of simple calculations we obtain

$$\begin{aligned} &\frac{\partial U}{\partial x} = 2\frac{U_0}{a}\cos\left(\frac{x}{a}\right), \qquad \frac{\partial U}{\partial t} = 0; \qquad \frac{\partial \Omega}{\partial t} = 0; \\ &\Omega = \frac{\partial U}{\partial x} = 2\frac{U_0}{a}\cos\left(\frac{x}{a}\right); \qquad \frac{\partial \Omega}{\partial x} = -2\frac{U_0}{a^2}\sin\left(\frac{x}{a}\right). \end{aligned} \qquad (3.102)$$

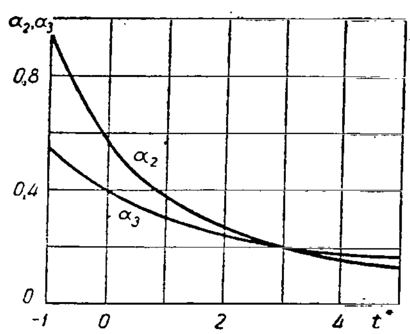



Fig. 29. The Coefficients $\alpha_1$ and $\alpha_2$ as a Function of the Suction Parameter $t^*$.

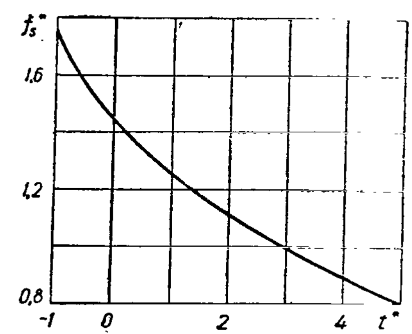



Fig. 30. The Shape Factor $f_s^*$ at the Separation Point of the Layer as a Function of the Suction Parameter $t^*$.

From the trailing critical point ($x = \pi a$) we derive the ordinary differential equation

$$\frac{df^*}{dt} = 4\frac{U_0}{a}(1 + 2\alpha_1 - \alpha_3) f^* - 4\frac{U_0}{a}(\alpha_2 + t^*). \qquad (3.103)$$

Using the initial condition $f^* = 0$ when $t = 0$, we obtain the solution to equation (3.103) in the form

$$f^* = \frac{\alpha_2 + t^*}{1 + 2\alpha_1 - \alpha_3}\left\{1 - \exp\left[4(1 + 2\alpha_1 - \alpha_3)\right]\frac{U_0 t}{a}\right]\right\} \qquad (3.104)$$

Since at the moment of separation formation of the boundary layer at the trailing critical point $t = t_s$, and $\sigma = U_0 t_s$, we obtain from expression (3.100) /109

$$\exp[4(1+2\alpha_1-\alpha_3)]\frac{\sigma}{a}=\frac{1+2\alpha_1+\frac{\alpha_3}{\alpha_2}t^*}{\alpha_3\left(1+\frac{1}{\alpha_2}t^*\right)} \qquad (3.105)$$

After determining the values of the coefficients $\alpha_1$, $\alpha_2$ and $\alpha_3$, as a function of the parameter $t^*$, we will determine from expression (3.105) the path followed by the circular cylinder before the moment when separation of the boundary layer begins and the eddy forms. Computations were made for negative values of $t^*$, corresponding to the presence of fluid suction from the boundary layer. From analyzing the results of these calculations, it follows that fluid suction from the boundary layer allows us to extend to a significant degree the path of the circular cylinder prior to the formation of the eddy at the trailing critical point.

For a nonpermeable surface, i.e., when $t^* = 0$, we find from equation (3.105) that $\sigma = 0.30a$. This value is practically no different from the value $\sigma = 0.32a$, calculated by the third approximation to the precise solution.

The method under discussion was developed for a plane boundary layer. With the help of the known transformation of Ye.I. Stepanov [48], it is easy to calculate a nonstationary laminar boundary layer with longitudinal streamlining of the body of revolution. In this case we must supplement the right-hand side of the integral impulse relationship (3.99) with the term $2\alpha_1\frac{1}{r_0}\cdot\frac{dr_0}{dx}Uf^*$, which takes into account the lateral curvature of the body of rotation.

# CHAPTER 4

## A THREE-DIMENSIONAL BOUNDARY LAYER

### A Boundary Layer and the Resistance of a Plate with Slip /110

We will now examine streamlining with the slip of a flat plate of infinite length in the presence of fluid suction from a laminar boundary layer by a uniform flow of viscous incompressible fluid [31]. The velocity vector of the external flow $U_0$ forms an angle $\beta$ (Fig. 31) with the leading edge. We will use a Cartesian coordinate system in connection with the plate. The $Ox$ axis is directed along the flow perpendicular to the leading edge, the $Oy$ axis is directed vertically perpendicular to the plane, and the $Oz$ axis is directed laterally parallel to the leading edge of the plate.

In this coordinate system the equations for a laminar boundary layer and the equation of continuity (1.16) for the case of a plate will have the form

$$\left.\begin{aligned} & u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2};\\ & u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}=\nu\frac{\partial^2 w}{\partial y^2};\\ & \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\end{aligned}\right\} \tag{4.1}$$

where $u$, $v$, $w$ are the longitudinal, lateral and transverse velocity components in the boundary layer.

In the presence of suction of a fluid across a nonpermeable surface of a plate, the following boundary conditions must be satisfied:

$$\begin{array}{lll} u=w=0; & v=-v_0(x) & \text{with } y=0; \\ u=U_0\cos\beta; & w=U_0\sin\beta & \text{with } y\to\infty. \end{array} \tag{4.2}$$

In the first boundary condition $v_0$ denotes the suction rate of a fluid across the surface of the plate. /111

Using the change of variables

$$u = U_0 \cos\beta \frac{df_0}{d\eta} \quad \eta = \frac{y}{\sqrt{\frac{U_0 \cos\beta}{\nu x}}}, \tag{4.3}$$

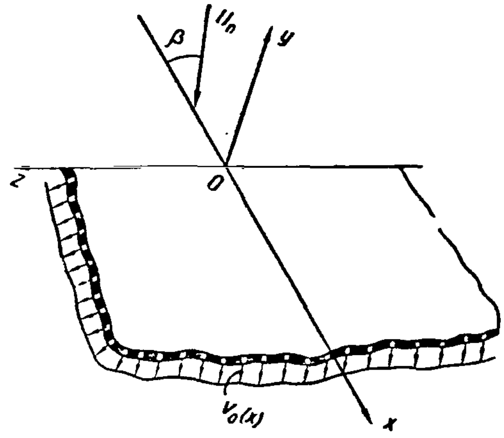



Fig. 31. A System of Coordinate Axes for Streamlining of a Plate with Slip.

we can reduce the first equation of system (4.1) to a known Blasius differential equation

$$\frac{d^3 f_0}{d\eta^3} + \frac{1}{2} \frac{d^2 f_0}{d\eta^2} f_0 = 0, \tag{4.4}$$

and we will convert the boundary conditions to the form

$$f_0 = C; \quad \frac{df_0}{d\eta} = 0 \ \text{with}\ \eta = 0;$$
$$\frac{df_0}{d\eta} \to \infty \qquad \text{with}\ \eta \to \infty. \tag{4.5}$$

With satisfaction of the /112 boundary conditions (4.5) equation (4.4) was integrated on a computer. Detailed tables of the functions $f_0(\eta)$ and its first, second and third derivatives were presented in reference [62] for different values of the constant $C$. From the third equation of system (4.1), the equation of continuity, we have

$$v = \frac{U_0 \cos\beta}{2\sqrt{R_x}} \left[ \eta \frac{df_0}{d\eta}(\eta) - f_0(\eta) \right]. \tag{4.6}$$

Using expression (4.6) and the first boundary condition (4.5), we can compute that

$$C = -\frac{2v_0}{U_0 \cos\beta} \sqrt{R_x}. \tag{4.7}$$

The following function satisfies the second equation of system (4.1)

$$w = \text{const}\, u. \tag{4.8}$$

Satisfying the second boundary condition of (4.2) we obtain

$$\text{const} = \tan\beta\,. \tag{4.9}$$

From equations (4.8) and (4.9), we find that

$$w = \frac{df_0}{d\eta} U_0 \sin\beta. \tag{4.10}$$

It follows from the last expression that the flow rate on the boundary layer in the presence of fluid suction across the surface of a plate agrees at all points with the direction of velocity of the external flow.

We will not determine the hydrodynamic forces acting on the permeable flat plate with slip and the presence of fluid suction from the boundary layer.

The coefficient of the tangential friction force is

$$C_{x_0} = \frac{2\nu}{U_0^2 x} \int_0^x \left(\frac{\partial u}{\partial y}\right)_{y=0} dx. \tag{4.11}$$

From expression (4.3) we compute that

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = U_0 \cos\beta \left(\frac{d^2 f}{d\eta^2}\right)_{\eta=0} \sqrt{\frac{U_0 \cos\beta}{\nu x}}. \tag{4.12}$$

Substituting expression (4.12) into (4.11), we find that /113

$$C_{x_0} = \frac{4 \left(\frac{d^2 f_0}{d\eta^2}\right)_{\eta=0} \cos^{\frac{3}{2}}\beta}{\sqrt{R_x}}, \tag{4.13}$$

where $R_x = \frac{U_0 x}{\nu}$ is the local Reynolds number.

The coefficient of the lateral friction force is

$$C_{z_0} = \frac{2\nu}{U_0^2 x} \int_0^x \left(\frac{\partial w}{\partial y}\right)_{y=0} dx. \tag{4.14}$$

From expression (4.10) we compute that

$$\left(\frac{\partial w}{\partial y}\right)_{y=0} = U_0 \sin\beta \left(\frac{d^2 f_0}{d\eta}\right)_{\eta=0} \sqrt{\frac{U_0 \cos\beta}{\nu x}}. \tag{4.15}$$

Substituting expression (4.15) into equation (4.14), we obtain that

$$C_{z_0} = \frac{4\left(\frac{d^2 f_0}{d\eta^2}\right)_{\eta=0} \sin\beta \cos^{1/2}\beta}{\sqrt{R_x}}. \qquad (4.16)$$

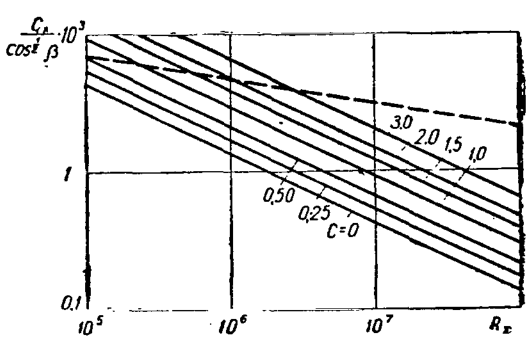
The value of the coefficient of total friction force is

$$C_x = C_{x_0}\cos\beta + C_{z_0}\sin\beta = \frac{4\left(\frac{\partial^2 f_0}{\partial\eta^2}\right)_{\eta=0} \cos^{1/2}\beta}{\sqrt{R_x}}. \qquad (4.17)$$

In Figure 32 the solid line corresponds to $\frac{C_x}{\cos^{\frac{1}{2}}\beta}$ as a function of the local Reynolds number $R_x$ for different values of the suction parameter $C$ (cf. expression (4.7) for a laminar boundary layer), and the broken line corresponds to a function for the coefficient of turbulent friction of the plate according to Prandtl-Schlichting taking the slip into account

$$\frac{C_x}{\cos^{1/2}\beta} = \frac{0,455}{(\lg \mathrm{Re})^{2,58}}. \qquad (4.18)$$

From an analysis of the data shown in the graph, it follows that with an increase in the Reynolds number the value of the coefficient /114
$\frac{C_x}{\cos^{\frac{1}{2}}\beta}$ changes equidistantly to the function of laminar friction according to Blasius ($C = 0$) and an increase in the suction intensity also leads to an increase in the resistance of laminar friction of the plate in the presence of slip. In the particular case of a non-permeable plate ($C = 0$), the results obtained agree with the solution of V.V. Struminskiy [50].




Comparison of the functions allows us to evaluate those limits in which suction of a laminar boundary layer can lead

Fig. 32. The Coefficient of Friction with Streamlining of a Plate With Slip as a Function of the Local Reynolds Number in the Presence of Suction.

to a decrease in frictional resistance. Since it is known that the profile drag of wings, streamlined with optimal angles of attack, is only 10-20% greater than the friction resistance of a plate, the obtained data (Fig. 32) can also be used for an approximate evaluation of the profile drag of the slip wings.

## The Boundary Layer of a Plate With Parabolic External Flow

No one has yet succeeded in obtaining the solution to the equations for a three-dimensional boundary layer, formed with the motion of a body of arbitrary shape, in its general form. In connection with this fact, we will limit ourselves to the integration of the equations for a three-dimensional boundary layer on a permeable surface of a plate with a parabolic external flow in order to clarify the basic picture illustrating the influence of fluid suction on /115
the nature of secondary flows in the layer. This approach allows us to obtain a precise solution to the problem and to analyze the possible influence of fluid suction across a permeable surface on the secondary flows in a three-dimensional boundary layer.

We will examine streamlining of a semi-infinite plate with a permeable surface in the presence of suction by a parabolic external flow [32, 30]. By a permeable plate we will mean a plate streamlined by a fluid, on whose surface there is only a non-zero normal velocity component, i.e., $w_0 = 0$ and $v_0 \neq 0$.

Later we will use a rectangular coordinate system, whose origin is located at the leading edge of the plate. The $x$ axis will be directed along its surface perpendicular to the leading edge, the $y$ axis will be normal to the plane of the plate and the $z$ axis will be perpendicular to the plane $xOy$. The velocity components for a parabolic external flow are

$$U = \text{const}; \quad V = 0; \quad W = a + bx, \tag{4.19}$$

where $a$ and $b$ are constant values.

In this case the streamlines of the external flow are parabolas and the current has a constant vorticity with a vorticity vector directed along the $y$ axis. The Helmholtz relationships for an eddy are satisfied. In addition, the velocities are not a function of the coordinate. Therefore, the system of equations (1.16) is significantly simplified:

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial p}{\partial y} &= 0; \\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} - bU &= \nu\frac{\partial^2 w}{\partial y}; \end{aligned}\right\} \tag{4.20}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

and the boundary conditions take the form /116

$$u=w=0; \quad v=-v_0 \text{ with } y=0;$$
$$u \to U; \quad w \to a+bx \text{ with } y \to \infty. \tag{4.21}$$

Using the change of variables

$$u=Uf_0'(\eta); \qquad \eta=y\sqrt{\frac{U}{\nu x}},$$

we can reduce the first equation of system (4.20) to a Blasius differential equation

$$f_0''' + \frac{1}{2} f_0'' f_0 = 0 \tag{4.22}$$

with the boundary conditions

$$f_0=C; \quad f_0'=0 \text{ with } \eta=0;$$
$$f_0' \to 1 \text{ with } \eta \to \infty. \tag{4.23}$$

Equation (4.22) with boundary conditions (4.23) was integrated on a computer by Emmons and Leigh. Detailed tables of the functions $f_0(\eta)$, $f_0'(\eta)$, $f_0''(\eta)$ and $f_0'''(\eta)$ were given in reference [62] values of $f(0)$ varying form 1.2385 to $\infty$.

From the equation of continuity of system (4.20) we obtain

$$v=\frac{1}{2}\cdot\frac{U}{x}\sqrt{\frac{\nu x}{U}}\,[f_0(0)+g(\eta)], \tag{4.24}$$

where

$$g(\eta)=\int_0^{\eta} \eta_1 f_0''(\eta_1)\,d\eta_1. \tag{4.25}$$

After integrating relationship (4.25) by parts, we obtain

$$\eta f_0' - g = f_0(\eta) - f_0(0). \tag{4.26}$$

If the velocity component parallel to the leading edge of the plate can be represented in the form

$$w = aw_0(\eta) + bxw_1(\eta), \tag{4.27}$$

then the third equation of system (4.20) can be given in this form: /117

$$\frac{a}{x}\left\{w_0'' + \frac{1}{2}[\eta f_0' - g - f_0'(0)]w_0'\right\} + \\ + b\left\{w_1'' + \frac{1}{2}[\eta f_0' - g - f_0(0)]w_1' - f_0'w_1 + 1\right\} = 0. \tag{4.28}$$

Using relationship (4.26) we convert equation (4.28):

$$\frac{a}{x}\left\{w_0'' + \frac{1}{2}f_0\left[1 - \frac{2f_0(0)}{f_0}\right]w_0'\right\} + b\left\{w_1'' + \frac{1}{2}f_0\left[1 - \right.\right. \\ \left.\left. - \frac{2f_0(0)}{f_0}\right]w_1' - f_0'w_1 + 1\right\} = 0. \tag{4.29}$$

Formula (4.29) is indentical over $x$ and consequently, the values of $w_0$ and $w_1$ must satisfy the following equations:

$$w_0'' + \frac{1}{2}f_0\left[1 - \frac{2f_0(0)}{f_0}\right]w_0' = 0; \tag{4.30}$$

$$w_1'' + \frac{1}{2}f_0\left[1 - \frac{2f_0(0)}{f_0}\right]w_1' + f_0'w_1 + 1 = 0 \tag{4.31}$$

with satisfaction of the boundary conditions

$$w_0 = 0; \quad w_1 = 0 \text{ with } \eta = 0; \\ w_0 \to 1; \quad w_1 \to 1 \text{ with } \eta \to \infty. \tag{4.32}$$

Equations (4.30) and (4.31) with boundary conditions (4.32) were integrated on a Minsk-1 computer for the following values of $f_0(0)$: 0; 0.5; 1.5; 2. The results are shown in Table 9. We should emphasize that for the particular case $f_0(0) = 0$ these results agree with an accuracy to three decimal places with the data of Loos [84].

It follows from equation (4.30) that with fluid suction from the boundary layer [$f_0(0) \neq 0$] the velocity component of the secondary flow $w_0$, caused by the constant velocity component in the leading flow $W = a$, can not have the same profile as the velocity component $u$. In other words, with a parallel leading flow, approaching

TABLE 9. THE FUNCTIONS $w_0(\eta)$ AND $w_1(\eta)$ /118

| $\eta$ | $f_0(0)$: 0 | 0,5 | 1,0 | 1,5 | 2,0 |
|---|---|---|---|---|---|
| | | | $w_0(\eta)$ | | |
| 0 | 0,0000 | 0,0000 | 0,0000 | — | — |
| 0,4 | 0,1038 | 0,2084 | 0,0016 | — | — |
| 0,8 | 0,2072 | 0,4751 | 0,0042 | — | — |
| 1,2 | 0,3096 | 0,8127 | 0,0084 | — | — |
| 1,6 | 0,4096 | 0,1232 | 0,0150 | — | — |
| 2,0 | 0,5055 | 0,1741 | 0,0252 | — | — |
| 2,4 | 0,5954 | 0,2339 | 0,0404 | — | — |
| 2,8 | 0,6772 | 0,3022 | 0,0621 | — | — |
| 3,2 | 0,7497 | 0,3773 | 0,0921 | — | — |
| 3,6 | 0,8116 | 0,4568 | 0,1320 | — | — |
| 4,0 | 0,8626 | 0,5379 | 0,1828 | — | — |
| 4,4 | 0,9030 | 0,6175 | 0,2451 | — | — |
| 4,8 | 0,9338 | 0,6925 | 0,3186 | — | — |
| 5,2 | 0,9565 | 0,7603 | 0,4018 | — | — |
| 5,6 | 0,9724 | 0,8194 | 0,4924 | — | — |
| 6,0 | 0,9831 | 0,8687 | 0,5870 | — | — |
| 6,4 | 0,9901 | 0,9083 | 0,6347 | — | — |
| 6,8 | 0,9944 | 0,9388 | 0,6821 | — | — |
| 7,2 | 0,9969 | 0,9614 | 0,7739 | — | — |
| 7,6 | 0,9984 | 0,9775 | 0,8591 | — | — |
| 8,0 | 0,9992 | 0,9884 | 0,9350 | — | — |
| 8,4 | 0,9996 | 0,9955 | 1,0000 | — | — |
| 8,8 | 0,9998 | 1,0000 | — | — | — |
| 9,2 | 0,9999 | — | — | — | — |
| 9,6 | 1,0000 | — | — | — | — |
| | | | $w_1(\eta)$ | | |
| 0 | 0,0000 | 0,0000 | 0,0000 | 0,0000 | 0,0000 |
| 0,4 | 0,4894 | 0,4034 | 0,3427 | 0,2971 | 0,2609 |
| 0,8 | 0,8270 | 0,6913 | 0,5972 | 0,5259 | 0,4684 |
| 1,2 | 1,0339 | 0,8773 | 0,7716 | 0,6915 | 0,6262 |
| 1,6 | 1,1385 | 0,9828 | 0,8817 | 0,8056 | 0,7427 |
| 2,0 | 1,1714 | 1,0318 | 0,9454 | 0,8811 | 0,8270 |
| 2,4 | 1,1610 | 1,0458 | 0,9787 | 0,9292 | 0,8867 |
| 2,8 | 1,1298 | 1,0420 | 0,9940 | 0,9591 | 0,9282 |
| 3,2 | 1,0935 | 1,0314 | 0,9999 | 0,9772 | 0,9562 |
| 3,6 | 1,0614 | 1,0205 | 1,0014 | 0,9877 | 0,9744 |
| 4,0 | 1,0369 | 1,0120 | 1,0013 | 0,9936 | 0,9857 |
| 4,4 | 1,0204 | 1,0064 | 1,0008 | 0,9969 | 0,9925 |
| 4,8 | 1,0104 | 1,0031 | 1,0004 | 0,9985 | 0,9962 |
| 5,2 | 1,0049 | 1,0014 | 1,0002 | 0,9993 | 0,9982 |
| 5,6 | 1,0021 | 1,0006 | 1,0001 | 0,9997 | 0,9992 |
| 6,0 | 1,0008 | 1,0002 | 1,0000 | 1,0000 | 0,9997 |
| 6,4 | 1,0003 | 1,0001 | — | — | — |
| 6,8 | 1,0001 | 1,0000 | — | — | — |
| 7,2 | 1,0000 | — | — | — | — |

the flat plate at an arbitrary angle, the direction of the flow in the presence of fluid suction from the boundary layer across a /119 permeable surface coincides with the flow direction in the leading flow. It is obvious that in a boundary layer of a permeable plate, lateral flows arise analogously to the case of flow without fluid suction from the boundary layer, but in the presence of a pressure gradient on the outer boundary of the layer.

In the particular case when the fluid is not sucked from the boundary layer [$f_0(0) = 0$], the function $w_0 = f'(\eta)$ satisfies equation (4.30) and boundary conditions (4.32). Then the velocity component of the secondary flow $w_0$ has the same profile as the velocity component of the longitudinal flow; the directions of the streamlines in the external flow coincide with those in the boundary layer. This phenomenon was first observed by Sears [106].

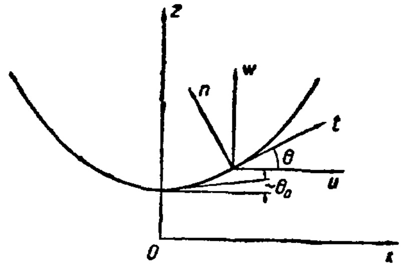


Fig. 33. A System of Coordinate Axes With Streamlining of a Plate by Parabolic External Flow.

We will now find on the one hand, the connection between $u$ and $w$, and on the other hand, that between the velocity components $t$ and $n$ along the tangent and normal to the local direction of the external flow (Fig. 33). We will examine in detail the case when the coefficient $a = 0$, i.e., when the streamlines are perpendicular to the leading edge of the plate. We will denote by $\Theta$ the angle between the direction of the external flow and the positive $x$ axis. Then

$$t = u\cos\Theta + w\sin\Theta; \tag{4.33}$$

$$n = w\cos\Theta - u\sin\Theta. \tag{4.34}$$ /120

Since $a = 0$, the function $u = Uf_0'(\eta)$ and $T = \sqrt{U^2 + b^2x^2}$, the local velocity of the external flow, then equations (4.33) and (4.34) can be written in the form

$$\frac{t}{T} = \frac{f_0'(\eta) + \left(\frac{bx}{U}\right)^2 w_1(\eta)}{1 + \left(\frac{bx}{U}\right)^2}; \tag{4.35}$$

$$\frac{n}{T} = \frac{bx}{U}\cdot\frac{w_1(\eta) - f_0'(\eta)}{1 + \left(\frac{bx}{U}\right)^2}. \tag{4.36}$$

Furthermore, we will express the velocity components $t$ and $n$ by the angle $\Theta$ for the external flow. Since

$$\tan\Theta = \frac{bx}{U}, \tag{4.37}$$

then

$$\frac{n}{T} = [w_1(\eta) - f_0'(\eta)]\frac{\sin 2\Theta}{2}, \tag{4.38}$$

$$\frac{t}{T} = f_0'(\eta)(1 - \sin^2\Theta) + w_1(\eta)\sin^2\Theta. \tag{4.39}$$

Using the results of computations of the functions $w_1(\eta)$ (cf. Table 9), we computed according to formulas (4.38) and (4.39) the profiles of the normal and tangential velocity components relative to the external flow with fluid suction from the boundary layer across the permeable surface of the plate. In these calculations we used the values of the functions computed by Emmons and Leigh [62]. On the basis of these data with the angle $\Theta$ equal to 0, 15, 30, 45, 60 and 90°, for the case of suction, the curves shown in Figure 34 were constructed.

With fluid suction from the boundary layer across a permeable surface of the plate, which is characterized by the value $f_0(0) = 1.0$ (Fig. 34a), the profile of the tangential velocity component varies comparatively little with the value of $\Theta$. In this case the velocity in the boundary layer $t/T$ transfers smoothly into the velocity of the external flow, and the local velocities in the boundary layer always remain lower than the velocity of the external flow. At the same time the normal velocity component (Fig. 34b) /121
does not exceed 6-7% of the velocity component of the external flow, and the velocity profile for this component has an extremal character. Consequently, fluid suction across the porous surface is an effective means of decreasing or eliminating the secondary flows in a three-dimensional boundary layer.

In the general case, when the coefficient $a \neq 0$, the normal and tangential velocity components must be calculated according to the following formulas [84]:

$$\frac{n}{T} = (\tan\Theta - \tan\Theta_0)\cos^2\Theta\,[w_1(\eta) - f_0'(\eta)]; \tag{4.40}$$

$$\frac{t}{T} = f_0'(\eta) + \frac{1}{2}\sin\Theta\,(\tan\Theta - \tan\Theta_0)\,[w_1(\eta) - f_0'(\eta)], \tag{4.41}$$

where $\Theta_0$ is the angle between the direction of the external flow and the $x$ axis at the leading edge (cf. Fig. 33).

## A Boundary Layer on a Slip Wing of Infinite Wing Aspect Ratio

The system of equations (1.16) with boundary conditions (1.17) in the general form describes a flow in a three-dimensional boundary layer in the presence of suction. We will now examine a stationary flow in the case when the longitudinal, lateral and transverse velocity components in the boundary layer $u$, $v$ and $w$ are functions in only two variables, $x$ and $y$. Then the system of equations (1.16) is simplified and takes the form

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\varrho}\cdot\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2};\qquad \frac{\partial p}{\partial y}=0, \tag{4.42}$$

This system must be solved with the boundary conditions

$$\begin{aligned} &u=w=0; \qquad v=-v_0 \text{ with } y=0;\\ &u\to U(x); \qquad w\to W(x) \text{ with } y\to\infty. \end{aligned} \tag{4.43}$$

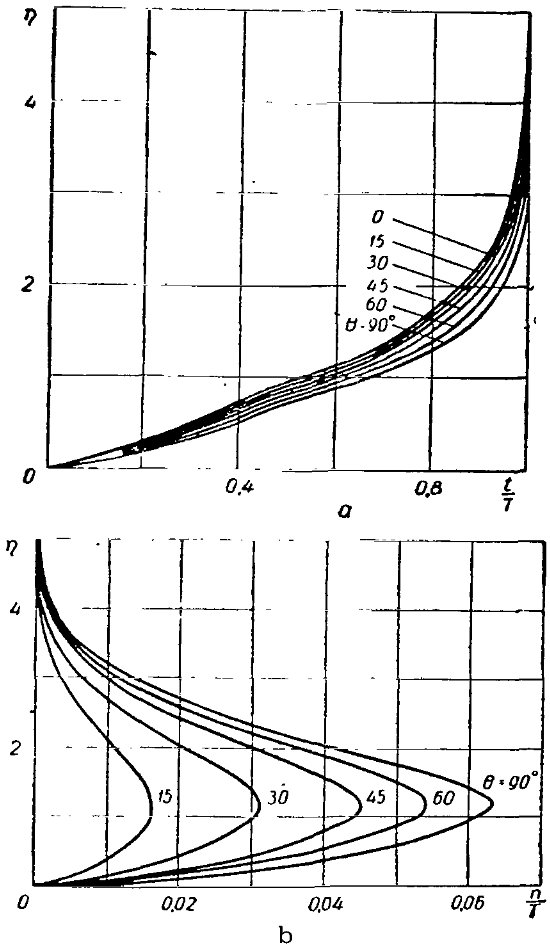
In boundary conditions (4.43), $U(x)$ and $W(x)$ denote the components of the velocity vector of the external flow on the coordinate axes $x$ and $z$ respectively. These components must satisfy the Euler equations, which in this case have the form

$$\begin{aligned} U\frac{dU}{dx}&=-\frac{1}{\varrho}\cdot\frac{\partial p}{\partial x};\\ U\frac{dW}{dx}&=-\frac{1}{\varrho}\cdot\frac{\partial p}{\partial z}. \end{aligned} \tag{4.44}$$

Taking relationship (4.44) into account, we can transform the system of equations (4.42):

$$\begin{gathered} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2};\\ \frac{\partial p}{\partial y}=0,\\ u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}=U\frac{dW}{dx}+\nu\frac{\partial^2 w}{\partial y^2};\qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \end{gathered} \tag{4.45}$$

The boundary conditions remain as before (4.43).




/122

Fig. 34. The Tangential (a) and Normal (b) Velocity Components in the Boundary Layer of a Plate in the Presence of Suction [$f(0) = 1.0$].

For the case (Fig. 35) of a boundary layer on a slip wing of infinite wing aspect ratio, $u(x)$ is defined by the streamlining of the wing in the plane perpendicular to its axis, $U(x)$ is the transverse velocity component on the outer boundary of the boundary layer and $\frac{\partial w}{\partial x} = 0$. Figure 36 shows the velocity profile in a three-dimensional boundary layer. /123

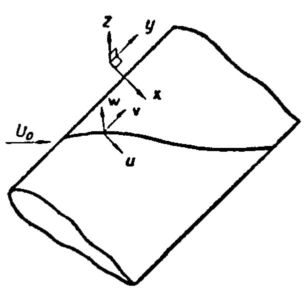



Fig. 35. The System of Coordinate Axes for a Wing of Infinite Wing Aspect Ratio with Slip.

The system of equations (4.45) with boundary conditions (4.43) in this case, as V.V. Struminskiy [50] first noted, can be broken down into two parts. The system

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2};$$

$$\frac{\partial p}{\partial y}=0;\qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{4.46}$$

corresponds to the problem discussed thoroughly in Chapters 2 and 3 /124 of this book. The remaining equation

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}=\nu\frac{\partial^2 w}{\partial y^2} \tag{4.47}$$

can serve as the basis for the following determination of the distribution of the transverse velocity component $w$ in a three-dimensional boundary layer.

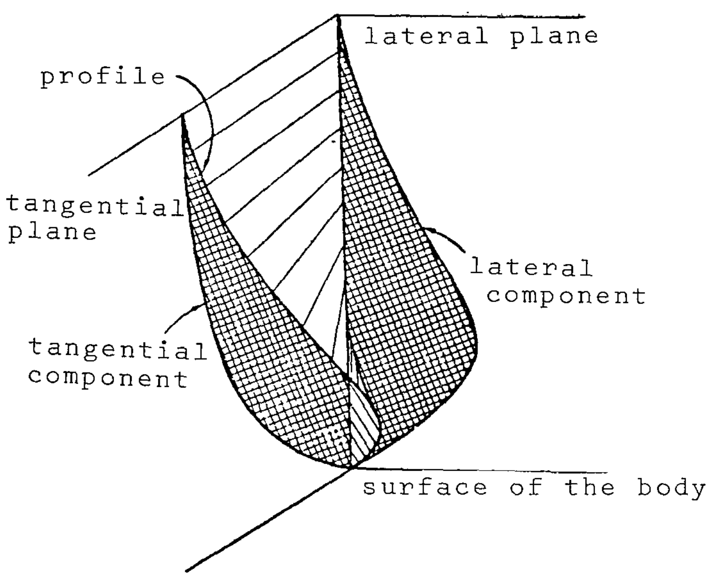



Fig. 36: The Velocity Profile in a Three-Dimensional Boundary Layer.

We will not transform the equation to an integral impulse relationship. In the general case, when the transverse velocity component on the outer boundary of the boundary layer $W = W(x)$,

$$\frac{\partial}{\partial x}[u(W-w)]+\frac{\partial}{\partial y}[v(W-w)]=\nu\frac{\partial^2 w}{\partial y^2}. \tag{4.48}$$

Integrating both sides of equation (4.48) across the layer over $y$ from zero to the thickness of the boundary layer $\delta_w$, determined by the change in $w$ from 0 to the value of $W(x)$, we obtain the unknown relationship

$$\frac{d\delta_w^{**}}{dx} + \frac{dU}{dx} \cdot \frac{\delta_w^{**}}{U} - \frac{v_0}{U} = \frac{\nu}{UW} \left( \frac{\partial w}{\partial y} \right)_{y=0}, \qquad (4.49)$$ /125

where $\delta_w^{**} = \int\limits_0^{\delta_w} \frac{u}{U}\left(1 - \frac{w}{W}\right) dy$ is the thickness of the impulse loss. In this relationship the term $v_0/U$ takes into account the presence of fluid suction from the boundary layer.

For the final derivation of the integral relationship (4.49) the following boundary conditions were used:

$$v = -v_0; \quad w = 0 \text{ with } y = 0$$

$$\frac{\partial w}{\partial y} = 0 \text{ with } y = \delta_w.$$

We now introduce into the examination the parameters

$$f = -\left[\frac{\partial^2 (u/U)}{\partial (y/\delta^{**})^2}\right]_{y=0}; \quad f_w = -\left[\frac{\partial^2 (w/W)}{\partial (y/\delta_w^{**})^2}\right]_{y=0}.$$

Using the system of equations in (4.46) and (4.47) we obtain

$$f = \frac{dU}{dx} \cdot \frac{\delta^{**2}}{\nu}; \quad f_w = 0.$$

We will now examine the dimensionless coefficient of friction

$$\zeta = \left(\frac{\partial u}{\partial y}\right)_{y=0} \frac{\delta^{**}}{U} = \left[\frac{\partial (u/U)}{\partial (y/\delta^{**})}\right]_{y=0} = \varphi'(0; f), \qquad (4.50)$$

where $\frac{u}{U} = \varphi\left(\frac{y}{\delta^{**}}; f\right)$ is a one-parameter profile set.

Since for a wing of infinite wing aspect ratio $\frac{dW}{dx} = 0$, when $f = 0$ the following equation must be satisfied

$$\varphi\left(\frac{y}{\delta^{**}}; 0\right) = \varphi\left(\frac{y}{\delta_w^{**}}\right),$$

where

$$\frac{w}{W} = \varphi\left(\frac{y}{\delta_w^{**}}, f_w\right).$$

Then for the first derivative we obtain /126

$$\left[\frac{\partial\left(\frac{w}{W}\right)}{\partial\left(\frac{y}{\delta_w^{**}}\right)}\right]_{y=0} = \psi'(0, 0) = \zeta(0). \qquad (4.51)$$

Stipulating $z_w = \frac{\delta_w^{**2}}{\nu}$, we transform the integral relationship (4.49)

$$\frac{dz_w}{dx} + 2\frac{dU}{dx}\cdot\frac{1}{U}z_w + \frac{1}{U}\cdot\frac{v_0\delta_w^{**}}{\nu} = \frac{2\zeta(0)}{U}, \tag{4.52}$$

where $\zeta(0) = \frac{a}{2} + (1-a)\frac{v_0\delta_w^{**}}{\nu}$, i.e., has the same value as in Chapter 3 and $a$ = 0.4408 = const.

We reduce the differential equation (4.52) to the form

$$\frac{dz_w}{dx} + 2\frac{dU}{dx}\cdot\frac{1}{U}z_w = U + \frac{(1-2a)}{U}\cdot\frac{v_0\delta_w^{**}}{\nu}. \tag{4.53}$$

As a result of the solution to equation (4.53) we obtain

$$z_w = \frac{1}{U^2(x)}\int_0^x \left[a - (1-2a)\frac{v_0\delta_w^{**}}{\nu}\right] U(x)\,dx. \tag{4.54}$$

The integration constant was taken to be zero form the condition of the finiteness of the value of $z_w(0)$ at the leading critical point.

Numerical integration of expression (4.54) is carried out by the method of successive approximations. In the first approximation we should take $v_0$ = 0. After computing with the required accuracy the shape factor $z_w$, we can determine the remaining characteristics of a laminar boundary layer in the presence of suction.

## An Approximate Calculation of the Axisymmetric and Three-Dimensional Boundary Layers. /127

The system of equations for an axisymmetric boundary layer on a body of revolution have the form [44]

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2};$$

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial y} = 0. \tag{4.55}$$

The system of coordinate axes and the basic notations are shown in Figure 37.

Since

$$r(x) = r_0(x) + y\cos\alpha, \tag{4.56}$$

where $r_0(x)$ is the instantaneous radius of the body of revolution throughout the boundary layer, except the area near the trailing

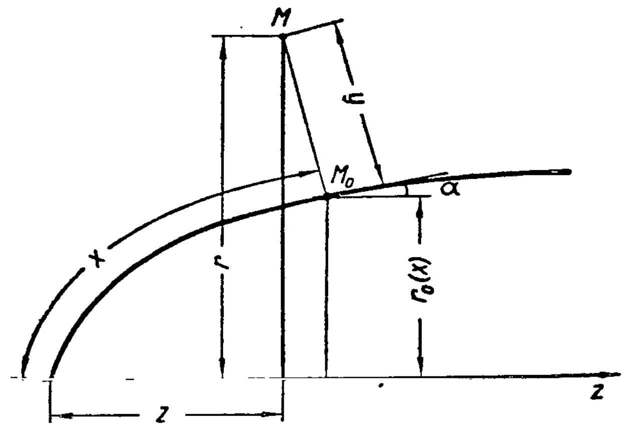



Fig. 37. The System of Coordinate Axes for an Axisymmetrical Boundary Layer on a Body of Revolution.

critical point, where the boundary layer is comparatively thick, and the radius of the curvature of the surface of the body $r_0$ tends to zero, in the equation of continuity of system (4.55) we can assume that $r(x) \approx r_0(x)$.

Using the system of equations (4.55) taking the observation /128
which was made into account, or applying the law of momentum change to an element of the axisymmetric boundary layer in a porous surface in the presence of suction, we obtain the integral impulse relationship [17]:

$$\frac{d}{dx}\int_0^\delta \varrho u^2 r_0 dy - U\frac{d}{dx}\int_0^\delta \varrho u r_0 dy - \varrho \dot{U}\frac{dU}{dx}\delta r_0 - \varrho v_0 r_0 U = -\tau_0 r_0. \qquad (4.57)$$

For the derivation of relationship (4.57) the origin of the coordinates was placed at the forward critical point, the $x$ axis was directed along the meridian of the axisymmetric body and the $y$ axis along the normal to the meridian. We should also note that relationship (4.57) and all the following operations are valid only for small values of $\delta/r_0$. Deviation from the last constraint significantly complicates the problem and does not allow us to obtain the solution in final form.

Introducing into the discussion the values

$$\delta^{**} = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy; \qquad \delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy; \qquad H = \frac{\delta^*}{\delta^{**}};$$

$$f = \frac{dU}{dx} \cdot \frac{\delta^{**2}}{\nu}; \qquad \zeta = \frac{\tau_0 \delta^{**}}{\mu U}; \quad t^{**} = \frac{v_0 \delta^{**}}{\nu}; \qquad F = 2\zeta - (2+H) f,$$

we convert relationship (4.57)

$$\frac{df}{dx} = \frac{dU}{dx} \cdot \frac{1}{U} (F - 2t^{**}) + \left( \frac{\frac{d^2U}{dx^2}}{\frac{dU}{dx}} - 2 \frac{1}{r_0} \cdot \frac{dr_0}{dx} \right) f. \qquad (4.58)$$

Equation (4.58) is a modification of the integral relationship (4.57), based on the assumption of the possibility of replacing the actual velocity profile in different sections of the boundary layer with a certain approximate one-parameter profile set. From an analogous relationship, given in reference [42], equation (4.58) differs only in the term $2t^{**}$, taking into account fluid suction across a porous surface.

Since in the axisymmetric case the function has the same form as for a plane boundary layer, according to formula (3.50) /129

$$F = A - Bf, \qquad (4.59)$$

where $A = 0.44 + 1.12t^{**}$; $B = 5.48$.

Integrating equation (4.58) taking (4.59) into account, we obtain

$$f(x) = \frac{dU}{dx} \cdot \frac{1}{U^B r_0^2} \int_0^x [(A(\xi) - 2t^{**}(\xi)] U^{1-B}(\xi) r_0^2(\xi) d\xi + C \frac{dU}{dx} \cdot \frac{1}{U^B r_0^2}. \qquad (4.60)$$

For a body with a nonpermeable surface ($t^{**} = 0$), expression (4.60) agrees with the solution obtained in reference [42]. Since the origin of the coordinates was chosen in such a way that $U = 0$ when $x = 0$, form the condition of the finiteness of the shape factor value $f(0)$ it follows that $C = 0$ and $f(0) = \frac{A}{B}$.

Calculation of the characterisitcs of an axisymmetric boundary layer on a porous surface in the presence of suction should be made using the method of successive approximations, analogously to the computation for a plane boundary layer. An axisymmetric boundary layer presents two problems. In the first problem values of $f(x)$

are computed according to the prescribed values of $r_0$, $U$ and $v_0$ according to formula (4.60) by the method of successive approximations. As a first approximation of the expected change in the parameter, we should take the corresponding values for a porous plate according to the data of Figure 26. Repeating the process of successive approximations, we can compute the unknown value with the necessary precision. Computations showed that for an elongated axisymmetric body ($\frac{L}{B}$ > 6) it is sufficient to limit ourselves to the third approximation.

After computing with the required accuracy the values of the shape factors $f$ and $t^{**}$, we can determine all the remaining characteristics of an axisymmetrical boundary layer:

$$\delta^{**} = \left[\frac{\nu f(x)}{\frac{dU}{dx}}\right]^{\frac{1}{2}}; \qquad \delta^* = \delta^{**}H; \tag{4.61}$$

$$\frac{\tau_0}{\frac{1}{2}\varrho U^2} = \frac{2\nu}{U\delta^{**}}\zeta; \qquad v_0 = \frac{\nu t^{**}}{\delta^{**}}.$$

The values of $H$ and $\zeta$ are computed by interpolation formulas (3.44) /130 and (3.46).

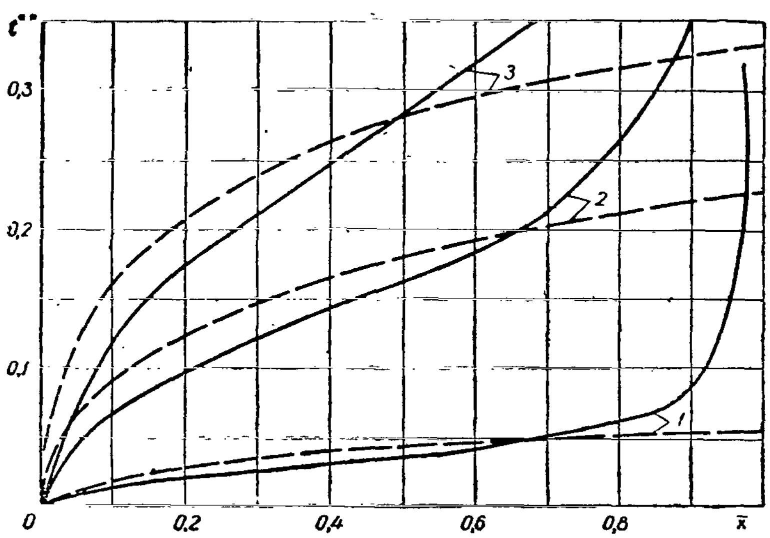



Fig. 38. Curves, Obtained as a Result of Calculations for an Axisymmetrical Laminar Boundary Layer on a Body of Revolution in the Presence of Suction: ···· First Approximation; ——— Second Approximation.

As an example, Figure 38 shows the curves obtained as a result of computation of the characteristics of a laminar boundary layer for an ellipse of revolution $\frac{L}{B} = 8$ with a porous surface with constant values of $\frac{v_0}{U}$ equal to $1.0 \cdot 10^{-4}$ (1); $0.5 \cdot 10^{-4}$ (2); $0.1 \cdot 10^{-4}$ (3) and the Reynolds number $\frac{U_0 B}{\nu} = \mathrm{Re}_B = 6.25 \cdot 10^6$.

This method can also be used to determine the characteristics of an axisymmetrical boundary layer and the function $v_0(x)$ according to the prescribed values of the functions $\delta^{**}$, $U$, $\frac{dU}{dx}$, $r_0$ and $\frac{dr_0}{dx}$. /131
If the enumerated values are given, then we can determine the function $f$ and $\frac{df}{dx}$ from equation (4.58) and then using formulas (4.61) compute the unknown functions $t^{**}$ and $v_0$. To compute the characteristics of an axisymmetrical boundary layer on a porous body of revolution in the presence of suction, we can also use the well known Stepanov-Mangler [48, 86] transformation.

We will use the integral impulse relationship (1.50) to calculate the characteristics of a three-dimensional boundary layer with steady motion. According to V.V. Struminskiy [49], we will assume that we can use as a velocity profile set in the cross sections of a nonstationary layer the profiles

$$\frac{u}{U} = \varphi\left(\frac{y}{\delta^{**}}; f\right),$$

where $\delta^{**} \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$ is the thickness of the impulse loss. In this expression the shape factor $f$ and the value of $\delta$ are functions of the coordinate $x$ and the time $t$. The assumption we have just made corresponds to the quasistationary process for examining phenomena in a nonstationary boundary layer. We note that in the case of a nonstationary boundary layer the shape factor is

$$f = \frac{\delta^{**2}}{\nu}\left(\frac{1}{U} \cdot \frac{dU}{dt} + \frac{dU}{dx}\right). \tag{4.62}$$

After the respective transformations taking the nonstationary term into account, we can reduce the integral relationship (1.50) to the form

$$\Phi(f, t^{**})\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} + \left[N(x, t) H'(f, t) + M(x, t) - 2\frac{\frac{d\Delta z}{dx}}{\Delta z}\right] f = \left(\frac{1}{U} \cdot \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x}\right)\left[F(f, t^{**}) - 2t^{**}\right]. \tag{4.63}$$

Here we introduce the following notations: /132

$$
\left.\begin{aligned}
&\Phi(f,t)=H(f,t^{**})+2f\frac{dH}{df};\\
&M(x,t)=\frac{\frac{\partial U}{\partial t}\cdot\frac{\partial U}{\partial x}+U\frac{\partial^2 U}{\partial x^2}+U\frac{\partial^2 U}{\partial t\partial x}}{U\frac{\partial U}{\partial x}+\frac{\partial U}{\partial t}}-4\frac{1}{U}\cdot\frac{\partial U}{\partial t};\\
&N(x,t)=\frac{\left(\frac{\partial U}{\partial t}\right)^2-U\frac{\partial^2 U}{\partial t^2}-U^2\frac{\partial^2 U}{\partial t\partial x}}{U\left(U\frac{\partial U}{\partial x}+\frac{\partial U}{\partial t}\right)};\\
&F(f,t^{**})=\{2\zeta(f,t^{**})-2\,[2-H(f,t^{**})]\,f\};\\
&\zeta=\frac{\tau_0\delta^{**}}{\mu U};\quad t^{**}=\frac{v_0\delta^{**}}{\nu};\quad H=\frac{\delta^*}{\delta^{**}}.
\end{aligned}\right\}\tag{4.64}
$$

Due to the assumption of quasistationarity, the values of the functions $\zeta$, $H$ and $F$ have the same form as for a stationary boundary layer. Therefore, these functions vary with time only through the shape factor $f$. In connection with this fact, in subsequent calculations for $H$ and $F$ we can use the interpolation formulas, obtained in reference [12]:

$$H=H_0-(2H_0-H_4)\,t^{**}-H_0\left(\frac{b}{a}-c\right)f;\tag{4.65}$$

$$F=A(t^{**})-B(t^{**})\,f.\tag{4.66}$$

Then we can write the function $\Phi(f,t^{**})$ as

$$\Phi(f,t^{**})=H_0-(2H_0-H_4)\,t^{**}-3H_0\left(\frac{b}{a}-c\right).\tag{4.67}$$

and transform equation (4.63) to the form

$$
\begin{aligned}
&[H_0-(2H_0-H_4)\,t^{**}-3H_0\left(\frac{b}{a}-c\right)f]\,\frac{\partial f}{\partial t}+U\frac{\partial f}{\partial x}+\\
&+N(x,t)\,[H_0-(2H_0-H_4)\,t^{**}-H_0\left(b-\frac{a}{c}\right)f]+M(x,t)\,f-\\
&-2\frac{1}{\Delta z}\cdot\frac{d\Delta z}{dx}\,f=\left(\frac{1}{U}\cdot\frac{\partial U}{\partial t}+\frac{\partial U}{\partial x}\right)\{[A\,(t^{**})-B\,(t^{**})\,f]-2t^{**}\},
\end{aligned}\tag{4.68}
$$

where $M(x,t)$; $\frac{1}{\Delta z}\cdot\frac{d\Delta z}{dx}$ and $N(x,t)$ are prescribed functions. /133

We will use this method for calculating the characteristics of a nonstationary laminar boundary layer in the presence of suction to the region of non-separated streamlining [33]. For the flow in the region of separation of the boundary layer, this method is not sufficiently precise. Here it is necessary to use more precise values for the functions $\Phi$, $F$, $H$ and $\zeta$ which significantly complicated the calculations.

With a stationary boundary layer the integral relationship (4.68) is significantly simplified and can be reduced to a Bernoulli equation

$$\frac{df}{dx}=\left(\frac{U''}{U'}-2\frac{1}{\Delta z}\cdot\frac{d\Delta z}{dx}\right)f+\frac{U''}{U'}[A(t^{**})-B(t^{**})f-2t^{**}],$$

where (4.69)

$$U'=\frac{dU}{dx};\quad U''=\frac{d^2U}{dx^2}.$$

We will give the integral of equation (4.69) in the form

$$f(x)=\frac{U'(x)}{[U(x)]^{B}\Delta z^2(x)}\left(\int_0^x\left[A-2t^{**}\right]U^{B-1}(\xi)\Delta z^2(\xi)\,d\xi+C\right), \tag{4.70}$$

where $C$ is an arbitrary constant.

We will locate the coordinate system in such a way that when $x = o$, $U = 0$. Then from the condition of the finiteness of the shape factor $f$ at the critical point (i.e., when $x = 0$) it follows that the constant $C = 0$, and $f(0) = \frac{A}{B}$.

After computing the values of the shape factor $f(x,t)$ according to formulas (4.61) we can determine all the remaining characteristics of a three-dimensional laminar boundary layer.

We should note that a more precise method for calculating the characteristics of a nonstationary three-dimensional boundary layer can be developed, using the displacement thickness as the characteristic linear dimension.

# CHAPTER 5

## A BOUNDARY LAYER WITH SLOT SUCTION

### The Wuest Method for Calculating the Boundary Layer of a Plate /134

The important methods for calculating a laminar boundary layer with suction of a fluid through lateral slots placed on the surface of the body are based on the assumption that suction does not change the pressure distribution on the outer boundary of the boundary layer.

The first approximate method for a laminar boundary layer on a plate with slot suction was proposed by Wuest [128]. The method was based on the use of the one-parameter profile set

$$u = UF(\eta, k), \tag{5.1}$$

where $k$ is the shape factor; $\eta = \frac{v}{\delta}$ , and also on the simultaneous use of the integral relationships for impulses and energy.

For the case of a plate these relationships have the form

$$\frac{d\delta^{**}}{dx} - \frac{v_0}{U\delta} = E; \tag{5.2}$$

$$\frac{d\delta^{***}}{dx} - \frac{v_0}{U} = \frac{\nu}{U\delta} D, \tag{5.3}$$

where $D = 2\int\limits_0^\infty \left(\frac{\partial \bar{u}}{\partial \eta}\right)^2 d\eta$ is the dimensionless dissipation function; $E = \left(\frac{\partial \bar{u}}{\partial \eta}\right)_{\eta=0}$ is the dimensionless tangential stress on the surface of the body; $\bar{u} = \frac{u}{U}$. Introducing the notations $\frac{U\delta}{\nu} = Z$ and $\frac{U}{\nu} = \xi$, after /135 solving equations (5.2) and (5.3) with respect to $\frac{dz}{d\xi}$ and $\frac{dk}{d\xi}$ we obtain

$$\frac{dZ}{d\xi} = -\frac{Ef_3' - Df_2' + \frac{v_0}{U} Z (f_3' - f_2')}{Z (f_3 f_2' - f_2 f_3')}; \tag{5.4}$$

$$\frac{dk}{d\xi} = \frac{Ef_3 - Df_2 + \frac{v_0}{U} Z (f_3 - f_2)}{Z^2 (f_3 f_2' - f_2 f_3')}, \tag{5.5}$$

where

$$f_1(Z) = \frac{\delta^*}{\delta}; \quad f_2(k) = \frac{\delta^{**}}{\delta}; \quad f_3(k) = \frac{\delta^{***}}{\delta}.$$

Excluding the parameter $\xi$ and integrating equations (5.4) and (5.5), we arrive at the expression

$$\frac{Z}{Z_0} = -\exp \int_{k_0}^{k} \frac{(Ef_3' - Df_2') + \frac{v_0}{U} Z \ (f_3' - f_2')}{(Ef_3 - Df_2) + \frac{v_0}{U} Z (f_3 - f_2)} dk. \tag{5.6}$$

As a velocity distribution we will use the profile set

$$\frac{u}{U} = 1 - e^{\eta} - F(k, \eta), \tag{5.7}$$

where

$$F(k, \eta) = k \left( 1 - e^{-\frac{\eta}{\sqrt{k}}} - \sin \frac{\pi \eta}{6 \sqrt{k}} \right) \text{ with } 0 < \eta < 3\sqrt{k};$$

$$F(k, \eta) = -ke^{-\frac{\eta}{\sqrt{k}}} \text{ with } \eta < 3\sqrt{k}.$$

This profile set is shown graphically in Figure 39. Table 10 shows the values of $f_1$, $f_2$, $f_3$, $D$, $E$ and their various combinations as a function of the parameter $k$.

The integral of (5.6) is computed separately for the region between the slots and in the region of the slots. In the first case, since the suction rate is identically equal to zero ($v_0 \equiv 0$), the integral of (5.6) takes the form /136

$$\frac{Z}{Z_0} = \exp \left\{ \int_0^k -\frac{Ef_3' - Df_2'}{Ef_3 - Df_2} dk \right\}. \tag{5.8}$$

TABLE 10. THE VALUES OF AUXILIARY FUNCTIONS FOR THE BOUNDARY LAYER OF A PLATE IN THE PRESENCE OF SLOT SUCTION.

| $k$ | $f_1$ | $f_2$ | $f_3$ | $D$ | $E$ |
|---|---|---|---|---|---|
| 0,0 | 1,000 | 0,5000 | 0,83333 | 1,0000 | 1,0000 |
| 0,1 | 1,00285 | 0,4965 | 0,82679 | 0,99093 | 0,84935 |
| 0,2 | 1,00806 | 0,4897 | 0,81289 | 0,97534 | 0,78695 |
| 0,3 | 1,01481 | 0,4816 | 0,79527 | 0,95718 | 0,73907 |
| 0,4 | 1,02280 | 0,4726 | 0,77381 | 0,93776 | 0,69870 |
| 0,5 | 1,03187 | 0,4630 | 0,75453 | 0,91781 | 0,66313 |
| 0,6 | 1,04189 | 0,4529 | 0,73251 | 0,89792 | 0,63098 |
| 0,7 | 1,05279 | 0,4425 | 0,70847 | 0,87817 | 0,60142 |
| 0,8 | 1,06450 | 0,4319 | 0,68605 | 0,85904 | 0,57389 |
| 0,9 | 1,07696 | 0,4210 | 0,66179 | 0,84040 | 0,54805 |
| 1,0 | 1,09014 | 0,49986 | 0,63662 | 0,82245 | 0,52360 |

| $f_3-f_2$ | $f'_2-f'_3$ | $Ef_3-Df_2$ | $Df'_2-Ef'_3$ | $f'_2f_3-f'_3f_2$ |
|---|---|---|---|---|
| 0,33333 | 0 | 0,33333 | 0 | 0 |
| 0,33029 | 0,0240 | 0,21024 | 0,0202 | 0,0035 |
| 0,32319 | 0,0640 | 0,16208 | 0,0362 | 0,095 |
| 0,31367 | 0,0890 | 0,12679 | 0,0484 | 0,0155 |
| 0,30118 | 0,1015 | 0,09747 | 0,0512 | 0,0195 |
| 0,29147 | 0,1110 | 0,07499 | 0,0496 | 0,0224 |
| 0,27407 | 0,1175 | 0,05553 | 0,0472 | 0,0248 |
| 0,26078 | 0,1234 | 0,03749 | 0,0451 | 0,0266 |
| 0,25385 | 0,1280 | 0,02270 | 0,0426 | 0,0280 |
| 0,24033 | 0,1331 | 0,00888 | 0,0409 | 0,0296 |
| 0,22609 | 0,1370 | 0,00376 | 0,0406 | 0,0312 |

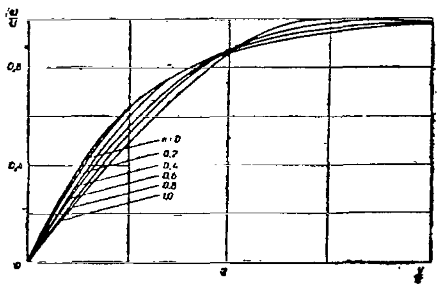



Fig. 39. The Set of Velocity Profiles in a Laminar Boundary Layer in the Presence of Slot Suction (According to Wuest).

Figure 40 shows $\frac{Z}{Z_0} = \frac{\delta}{\delta_0}$ as a function of the parameter $k$. The thickness of the boundary layer at the beginning of the slot is $\delta_0$.

Numerically integrating differential equation (5.5) we find $k$ as a function of $\xi = \frac{Ux}{\nu}$ and then $Z$ as a function of $\xi$ (Fig. 41). /137

In the region of the slot we use the velocity profile

$$\frac{u}{U} = 1 - e^{-\eta} - F(k_1, \eta) + F(k, \eta). \quad (5.9)$$

The function $F(k,\eta)$ has the form of (5.7) where at the beginning of the slot $k = k_1$. The suction rate is constant. The values of $f_2$; $\frac{df_2}{dk}$; $f_3$ and $\frac{df_3}{dk}$ as a function of the values of the parameters $k$ and

$k_1$ are shown in Figures 42 and 43.

For the dissipation and tangential stress functions $D$ and $E$ we obtain

$$D = D_{k=0} + 0{,}13587k^{\frac{3}{2}} + 4\left(\frac{k}{1+\sqrt{k}} - \frac{kk_1}{\sqrt{k_1}+\sqrt{k}}\right) -$$

$$-\frac{2\pi}{3}\cos\sqrt{\frac{k}{k_1}}\cdot\frac{\pi}{2}\cdot\frac{kk_1^{\frac{3}{2}}}{k_1-k} + \frac{2\pi}{3}kk_1\frac{\frac{\pi}{6}\sqrt{k_1}e^{-3\sqrt{\frac{k}{k_1}}} + \sqrt{k}}{k + k_1\frac{\pi^2}{36}} + \tag{5.10}$$

$$+\frac{2\pi}{3}kk_1\frac{\frac{\pi}{6}\sqrt{k}e^{-3\sqrt{\frac{k_1}{k}}} + \sqrt{k_1}}{k_1 + k\frac{\pi^2}{36}} - \frac{2\pi}{3}\cdot\frac{k\frac{\pi}{6}e^{-3\sqrt{k}} + \sqrt{k}}{k + \frac{\pi^2}{36}};$$ /138

$$E = 1 - 0{,}4764\left(\sqrt{k_1} - \sqrt{k}\right).$$

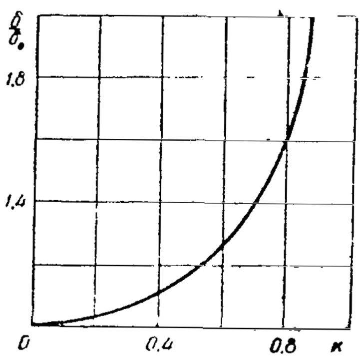



Fig. 40. The Thickness of the Boundary Layer as a Function of the Parameter $K$.

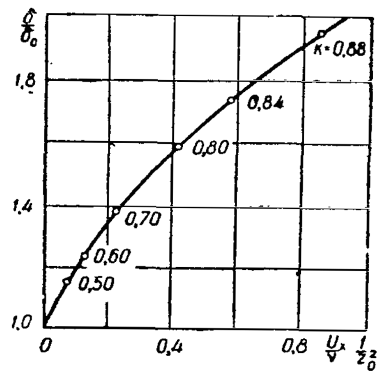



Fig. 41. The Thickness of the Boundary Layer as a Function of the Parameter $\frac{Ux}{\nu}\cdot\frac{1}{Z_0^2}$.

We will divide the calculation of the boundary layer in the region of the slot into two stages. The first stage corresponds to the change in the values of $k$ from 0 to $k_1$. In the second stage we will assume that the thickness of the boundary layer varies under the condition $k = k_1$ = const. Assuming that $k$, $f_2$ and $E$ are constant values, we obtain from the impulse equation the differential equation

$$\frac{1}{4}\cdot\frac{d(U^2Z_\infty^2)}{d\xi} = \left(\frac{v_0}{U}\right)^2\left(1 - \frac{Z}{Z_\infty}\right). \tag{5.11}$$

The integral of equation (5.11) is

$$\xi - \xi_c = \frac{1}{2}\left(\frac{U}{v_0}\right)^2\left(\frac{Z_c}{Z_\infty} + \ln\frac{Z_c - Z}{Z - Z_\infty}\right). \quad (5.12)$$

However, as the author himself agrees, this method is not perfected.

### The Lachmann Method and His Generalization for a Longitudinal Pressure Gradient on the Outer Boundary of the Layer /139

Using geometric constructions, G. Lachmann [82] proposed a very approximate method for calculating the thickness of the impulse loss for a laminar boundary layer in the presence of suction through lateral slots. This method is widely used by Soviet investigators [1]. For calculation in the region between the slots we use the known expression

$$\left(\frac{\delta^{**}}{L}\right)^2 = \frac{1}{\mathrm{Re}}\cdot\frac{A}{(U/U_0)^B}\times$$
$$\times\left(\int\limits_{\frac{x_{n-1}}{L}}^{\frac{x_n}{L}}\left(\frac{U}{U_0}\right)^{B-1}dx + \left|\frac{\left(\frac{\delta^{**}}{L}\right)^2\mathrm{Re}\left(\frac{U}{U_0}\right)^B}{A}\right|\right)_{x=x_{n-1}}, \quad (5.13)$$

where $A$ = 0.44; $B$ = 5.48.

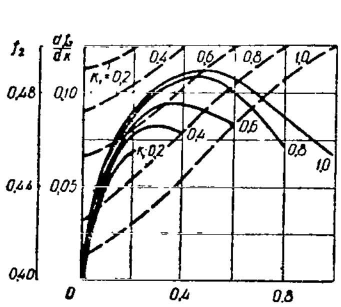



Fig. 42. Curves of the Values of the Function $f_2$ (the Broken Line) and its Derivative $\frac{df_2}{dk}$ (the Solid Line) in the Region of the Slot.

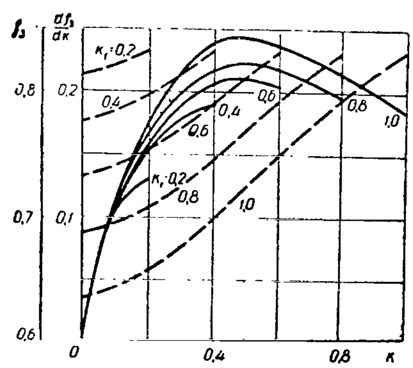



Fig. 43. Curves of the Values of the Functions $f_3$ (the Broken Line) and its Derivative $\frac{df_3}{dk}$ (the Solid Line) in the Region of the Slot.

Let $\left(\frac{\delta^{**}}{L}\right)_n$ be the ratio of the thickness of the impulse loss /140
to the length of the body upon reaching the $n$-th slot (Fig. 44). Because of suction this ratio decreases to the value $\left(\frac{\delta^{**}}{L}\right)'_n$. In the process the development of the boundary layer behind the slot the thickness of the impulse loss increases again and reaches the value $\left(\frac{\delta^{**}}{L}\right)_{n+1}$ at the $(n + 1)$-th slot.

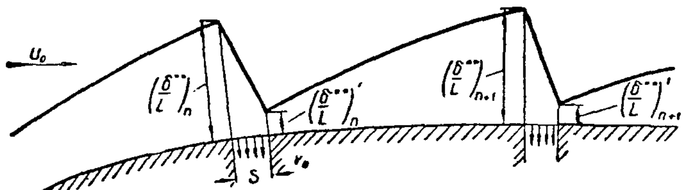



Fig. 44. A Diagram of the Change in the Thickness of the Impulse Loss Along the Contour of a Body in the Presence of Slot Suction of a Boundary Layer.

According to Lachmann, we will limit the value of the critical Reynolds number, defined by the thickness of the impulse loss, to the value $R_{\delta^{**}_{cr}} = 1250$.

From the functions

$$\left(\frac{\delta^{**}}{L}\right)_n = \frac{R_{\delta^{**}_{cr}}}{\mathrm{Re}\left(\frac{U}{U_0}\right)_n}; \tag{5.14}$$

$$\left(\frac{\delta^{**}}{L}\right)_{n+1} = \frac{R_{\delta^{**}_{cr}}}{\mathrm{Re}\left(\frac{U}{U_0}\right)_{n+1}} \tag{5.15}$$

we obtain the relationship

$$\frac{(\delta^{**}/L)'_n}{(\delta^{**}/L)_n} = \frac{\sqrt{\left[\left(\frac{U}{U_0}\right)^4_{n+1} - C_n \Delta_{n+1} l\right]}}{\left(\frac{U}{U_0}\right)^2_n}, \tag{5.16}$$

where

$$\Delta_{n+1}I=\int\limits_{(x/L)_n}^{(x/L)_{n+1}}\left(\frac{U}{U_0}\right)^{4,48}d\left(\frac{x}{L}\right);$$ /141

$$C_n=\frac{0{,}44\ \mathrm{Re}}{(R_{\delta^{**}})^2_{cr}}.$$

It now remains for us to find the relationship between the suction coefficient $C_Q = \frac{Q}{U_0 L}$ and the ratio of the thickness of impulse loss $(\delta^{**}/L)_n/(\delta^{**}/L)'_n$. According to the diagram (Fig. 45) we assume that after suction, having discarded the lower portion of profile *I* at a distance $y$ from the surface, we obtain profile *II*. The area of the discarded portion of the profile corresponds to the quantity of sucked-off fluid. Since the width of the slot $s$ is small, in our first approximation we can ignore the change in profile shape. We should emphasize that such an assumption is not obvious.

Since the discharge of sucked-off fluid through the slot is

$$Q=\int\limits_0^y u\,dy=U\delta^{**}\int\limits_0^{y/\delta^{**}}\frac{u}{U}\,d\left(\frac{y}{\delta^{**}}\right),$$

the coefficient of discharge is

$$C_Q=\frac{\delta^{**}U}{U_0L}\int\limits_0^{y/\delta^{**}}\frac{u}{U}\,d\left(\frac{y}{\delta^{**}}\right). \qquad (5.17)$$

For any slot we can write relationship (5.17) in the form

$$\frac{C_{Q_n}}{(\delta^{**}/L)_n}=\frac{U}{U_0}\int\limits_0^{y/\delta^{**}}\frac{u}{U}\,d\left(\frac{y}{\delta^{**}}\right); \qquad (5.18)$$

$$\frac{(\delta^{**}/L)'_n}{(\delta^{**}/L)_n}=1-\int\limits_0^{y/\delta^{**}}\frac{u}{U}\left(1-\frac{u}{U}\right)d\left(\frac{y}{\delta^{**}}\right). \qquad (5.19)$$

Equations (5.18) and (5.19) in parametric form determine the relationship of $\frac{C_{Q_n}}{\left(\frac{\delta^{**}}{L}\right)_n}$ to $\frac{\left(\frac{\delta^{**}}{L}\right)'_n}{\left(\frac{\delta^{**}}{L}\right)_n}$. This function was computed by /142

Lachmann (Fig. 46) on the assumption that for an approximation of the velocity profile across the boundary layer we can use a Blasius profile.

W. Colemann in reference [58] made an attempt to validate theoretically the Lachmann approximate method for calculating the characteristics of a laminar boundary layer with slot suction.

We will now examine the flow in the boundary layer near the slot (Fig. 47). The streamline *AB* passes at a certain distance $y_s$ from the surface. Point *B* is the critical point on the trailing

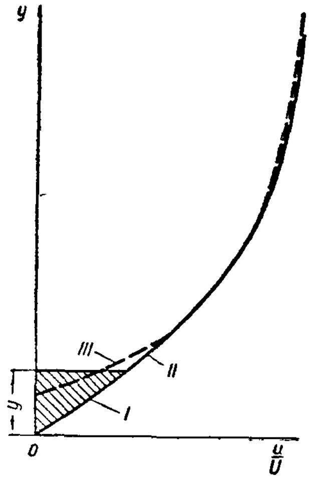



Fig. 45. A Diagram of the Change of the Profile with Slot Suction of a Laminar Boundary Layer (According to Lachmann): *I* is the Discarded Part of the Profile; *II* is the Actual Profile in Front of the Slot; *III* is the Actual Velocity Profile Behind the Slot.

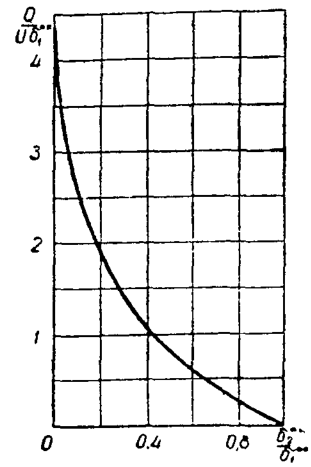



Fig. 46. The Coefficient of Discharge as a Function of the Ratio of Thicknesses of Impulse Loss in Front of and Behind the Slot (According to Lachmann).

edge of the slot. Consequently, the fluid flowing between streamline *AB* and the surface is completely sucked through the slot, and the other portion remains part of the boundary layer. We will examine cross sections 1 and 2, which are placed directly on the leading and trailing edges of the slot and we will calculate the dissipation energy of the flow, and also the change in momentum in the region of the slot. /143

We will position the $x$ coordinate axis along the $y$ axis normal to the surface of the body. We will write the equation of momentum in the form

$$\frac{dM}{dx} + \varrho U \frac{dU}{dx} \delta^* = \tau_0, \qquad (5.20)$$

where $M = \varrho \int_0^\infty u(U - u)\,dy$ is the momentum and $u$ is the longitudinal velocity component in the boundary layer.

We will assume that the width of the slot is small enough that in the interval $x_1 \leq x \leq x_2$, where $x_2 - x_1 = s$ (the width of the

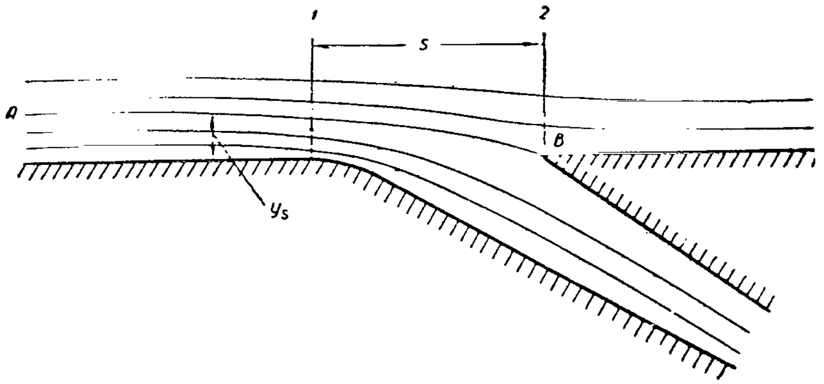



Fig. 47. Diagram of Slot Suction of a Boundary Layer (According to Colemann).

slot), the thickness of the displacement flow and the velocity on the outer boundary are

$$\delta^* = a_0 + a_1 x; \tag{5.21}$$

$$U = b_0 - b_1 x. \tag{5.22}$$

Since in the region of the slot the flow does not come in contact with the hard surface, Colemann assumes that $\tau_0 = 0$. Then we can write equation (5.20) in the form

$$\frac{dM}{dx} = \varrho b_1 \int_0^s [a_0 b_0 + (a_1 b_0 - a_0 b_1)\,x]\,dx = \varrho a_0 b_0 b_1 s \left[1 + \frac{1}{2}\left(\frac{a_1}{a_0} - \frac{b_1}{b_0}\right) s\right]. \tag{5.23}$$ /144

Practically speaking in the majority of cases the value

$$\frac{1}{2}\left(\frac{a_1}{a_0} - \frac{b_1}{b_0}\right) s \ll 1. \tag{5.24}$$

Then we can convert equation (5.23) to the form

$$M_2 - M_1 = \varrho\delta_1^* U_1 (U_1 - U_2), \tag{5.25}$$

where the indices 1 and 2 refer to the respective cross sections, and we can ignore the change in momentum due to friction resistence in the region of a very narrow slot ($U_2 \approx U_1$).

Thus in cross section 1

$$M_1 = \varrho\int_0^\infty u_1(U_1 - u_1)\,dy = \varrho\int_0^{y_s} u_1(U_1 - u_1)\,dy + \varrho\int_{y_s}^\infty u_1(U_1 - u_1)\,dy, \tag{5.26}$$

and in cross section 2

$$M_2 = \varrho\int_{y_s}^\infty u_1(U_1 - u_1)\,dy. \tag{5.27}$$

From this, the change in momentum is

$$M_1 - M_2 = \varrho\int_0^{y_s} u_1(U_1 - u_1)\,dy. \tag{5.28}$$

Since

$$\delta_1^{**} = \frac{M_1}{\varrho U_1^2}; \qquad \delta_2^{**} = \frac{M_2}{\varrho U_2^2},$$

and

$$U_2 \approx U_1,$$

then

$$\frac{\delta_2^{**}}{\delta_1^{**}} = 1 - \int_0^{y_s} \frac{u_1}{U_1}\left(1 - \frac{u_1}{U_1}\right) d\left(\frac{y}{\delta^{**}}\right). \tag{5.29}$$

Then the quantity of sucked-off fluid on a unit of length of the slot is /145

$$Q = \int_0^{y_s} u_1 dy_1 .$$

or in dimensionless form

$$C_Q = \frac{Q}{U_0 L} = \frac{U_1}{U_0} \cdot \frac{\delta_1^{**}}{L} \int\limits_0^{y_1/\delta^{**}} \frac{u_1}{U_1}\, d\left(\frac{y}{\delta^{**}}\right). \tag{5.30}$$

where $U_0$ is the velocity of the leading flow; $L$ is the length of the body.

Equations (5.29) and (5.30) are parametric equations, derived by Lachmann from certain other, purely geometric considerations.

To compute $C_Q$ as a function of $\frac{\delta_2^{**}}{\delta_1^{**}}$, Lachmann in reference [82] used a Blasius profile. To allow for the longitudinal velocity gradient on the outer boundary of the boundary layer when computing $C_Q$ as a function of $\frac{\delta_2^{**}}{\delta_1^{**}}$ by formulas (5.29) and (5.30), Colemann proposed using a Hartree profile. However, these computations can be made more simple using analytic velocity profiles. A comparison of precise and approximate results showed that the most reasonable function was the profile set proposed by A.M. Basin [2]:

$$\frac{u}{U} = \left\{1 + \frac{2}{\pi^2} f \frac{1}{H^{**2}} \left[1 - \sin\frac{\pi}{2} H^{**}\left(\frac{y}{\delta^{**}}\right)\right] \sin\frac{\pi}{2} H^{**}\left(\frac{y}{\delta^{**}}\right)\right\}, \tag{5.31}$$

where $H^{**} = \frac{\delta^{**}}{\delta}$ is the ratio of the conventional thicknesses of the boundary layer.

Substituting formula (5.31) into equation (5.29), after simple transformations we obtain

$$\begin{aligned} \frac{\delta_{cr}^{**}}{\delta_H^{**}} = 1 - \int\limits_0^{y/\delta^{**}} \left\{1 + \frac{2}{\pi^2} f \frac{1}{H^{**2}} \left[1 - \sin\frac{\pi}{2} H^{**}\left(\frac{y}{\delta^{**}}\right)\right]\right\} \times \\ \times \sin\left[\frac{\pi}{2} H^{**}\left(\frac{y}{\delta^{**}}\right)\right] d\left(\frac{y}{\delta^{**}}\right) + \int\limits_0^{y/\delta^{**}} \left\{1 + \frac{2}{\pi^2} f \frac{1}{H^{**2}} \times \right. \\ \left. \times \left[1 - \sin\frac{\pi}{2} H^{**}\left(\frac{y}{\delta^{**}}\right)\right]\right\} \sin^2\left[\frac{\pi}{2} H^{**}\left(\frac{y}{\delta^{**}}\right)\right] d\left(\frac{y}{\delta^{**}}\right), \end{aligned} \tag{5.32}$$

where $H^{**}$ varies only with the shape factor $f$. /146

After computing the integrals and reducing the terms equation (5.32) takes the form

$$\frac{\delta^{**}_{\mathrm{cr}}}{\delta^{**}_{H}} = 1 - a_1 + a_2\left(\frac{y}{\delta^{**}}\right) - a_3 \sin\left[\pi H^{**}\left(\frac{y}{\delta^{**}}\right)\right] + a_4 \sin\left[2\pi H^{**}\left(\frac{y}{\delta^{**}}\right)\right] + a_5 \cos\left[\frac{\pi}{2} H^{**}\left(\frac{y}{\delta^{**}}\right)\right] - a_6 \cos^3\left[\frac{\pi}{2} H^{**}\left(\frac{y}{\delta^{**}}\right)\right], \qquad (5.33)$$

where

$$a_1 = \frac{2}{\pi H^{**}} + \frac{20}{3} f \frac{1}{H^{**3}} + f^2 \frac{32}{3\pi^6} \cdot \frac{1}{H^{**6}};$$

$$a_2 = \frac{1}{2} + f \frac{3}{\pi^2} \cdot \frac{1}{H^{**2}} + f^2 \frac{7}{2\pi^4} \cdot \frac{1}{H^{**4}};$$

$$a_3 = \frac{1}{2\pi H^{**}} + f \frac{3}{\pi^3} \cdot \frac{1}{H^{**3}} + f^2 \frac{4}{\pi^5} \cdot \frac{1}{H^{**5}};$$

$$a_4 = f^2 \frac{1}{4\pi^5} \cdot \frac{1}{H^{**5}};$$

$$a_5 = \frac{2}{\pi H^{**}} + f \frac{12}{\pi^3} \cdot \frac{1}{H^{**3}} + f^2 \frac{16}{\pi^5} \cdot \frac{1}{H^{**5}};$$

$$a_6 = f \frac{8}{3\pi} \cdot \frac{1}{H^{**3}} + f^2 \frac{16}{3\pi^5} \cdot \frac{1}{H^{**5}}$$

As a result of analogous computations formula (5.30) can be reduced to the form

$$\frac{C_Q}{\delta^{**}_{\mathrm{cr}} \frac{U}{U_0}} = a_7 - a_8\left(\frac{y}{\delta^{**}}\right) + a_9 \sin\left[\pi H^{**}\left(\frac{y}{\delta^{**}}\right)\right] - a_{10} \cos\left[\frac{\pi}{2} H^{**}\left(\frac{y}{\delta^{**}}\right)\right], \qquad (5.34)$$

where /147

$$a_7 = \frac{2}{\pi H^{**}} + f \frac{4}{\pi^3} \cdot \frac{1}{H^{**3}};$$

$$a_8 = f \frac{1}{\pi^2} \cdot \frac{1}{H^{**2}};$$

$$a_9 = f \frac{1}{\pi^3} \cdot \frac{1}{H^{**5}};$$

$$a_{10} = \frac{2}{\pi H^{**}} + f \frac{4}{\pi^3} \cdot \frac{1}{H^{**3}}.$$

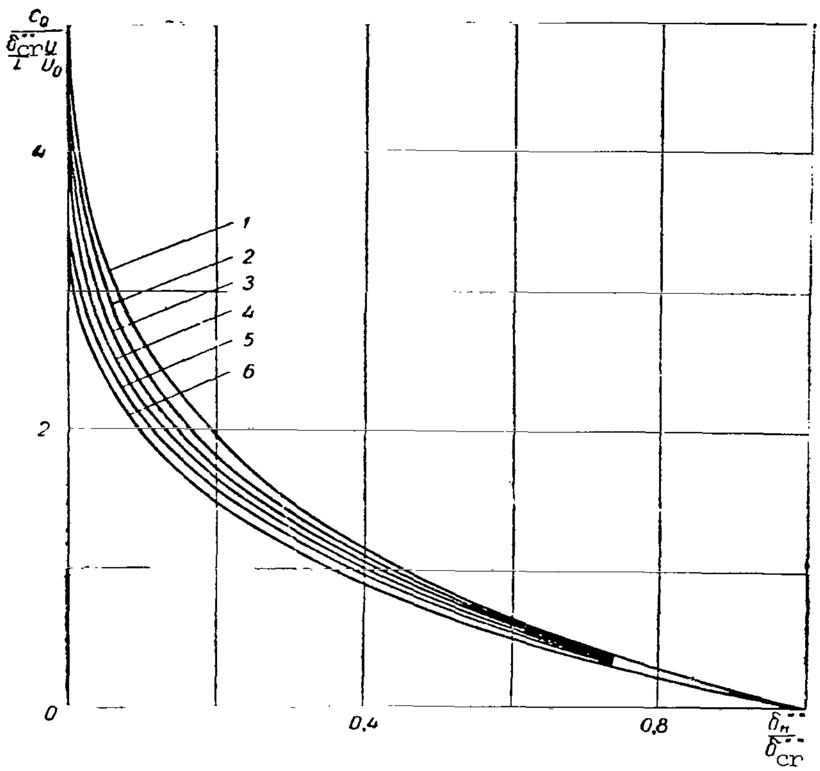



Fig. 48. The Dimensionless Discharge of Sucked-Off Fluid as a Function of the Ratio of the Thicknesses of Impulse Loss in Front of and Behind the Slot for the Velocity Profiles of A.M. Basin: 1. $f = 0.0453$; 2. $f = 0.01735$; 3. $f = 0$; 4. $f = -0.03776$; 6. $f = -0.07713$.

Equations (5.33) and (5.34) in parametric form express $\dfrac{C_Q}{\dfrac{\delta^{**}_{cr}}{L}\cdot\dfrac{U}{U_0}}$ /148

as a function of $\delta^{**}_H/\delta^{**}_{cr}$. The numerical values of the coefficients are shown in Table 11. As a result of the final calculations, we construced curves (Fig. 48) of $\dfrac{C_Q}{\dfrac{\delta^{**}_{cr}}{L}\cdot\dfrac{U}{U_0}}$ as a function of $\delta^{**}_H/\delta^{**}_{cr}$ for different values of the shape factor $f$. The data for $f = 0$ agree satisfactorily with Lachmann's results.

## Periodic Suction of a Boundary Layer

In reference [130] periodic suction of a boundary layer on a plate was investigated theoretically. We will now examine the case, interesting for engineering, of fluid suction through lateral slots placed at identical distances from one another.

TABLE 11. THE NUMERICAL VALUES OF THE COEFFICIENTS $H^{**}$ AND $a_i$ AS A FUNCTION OF $f$.

| $f$ | $H^{**}$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ | $a_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0,0453 | 0,0958 | 31,0182 | 2,8756 | 9,9704 | 0,2077 | 39,8760 | 8,8631 | 13,2912 | 0,5001 | 1,6616 | 13,2919 |
| 0,03164 | 0,1266 | 10,7950 | 1,2396 | 3,1160 | 0,0251 | 12,6724 | 1,8772 | 7,0408 | 0,06367 | 0,5024 | 7,0408 |
| 0,01735 | 0,1326 | 7,2959 | 0,8348 | 2,0159 | 0,0060 | 8,0635 | 0,7676 | 5,7609 | 0,1000 | 0,2400 | 5,7609 |
| 0 | 0,1366 | 4,561 | 0,5000 | 1,162 | 0 | 4,561 | 0 | 4,551 | 0 | 0 | 4,561 |
| −01893 | 0,1385 | 2,6968 | 0,2350 | 0,5516 | 0,0057 | 2,2065 | −0,4903 | 3,6773 | 0,1000 | −0,2298 | 3,6773 |
| −03776 | 0,1383 | 1,2893 | 0,04000 | 0,1376 | 0,0230 | 0,5503 | −0,7368 | 2,7618 | −0,2000 | −0,4603 | 2,7618 |
| −0,5529 | 0,1359 | 0,3523 | −0,0880 | −0,0983 | 0,0539 | −0,3930 | −0,7452 | 1,8426 | −0,3033 | −0,7105 | 1,8426 |
| −0,6826 | 0,1315 | −0,0648 | −0,1400 | −0,1453 | 0,0968 | −0,5809 | −0,5167 | 0,9687 | −0,3999 | −0,9681 | 0,9687 |
| −0,07413 | 0,1250 | −0,0081 | −0,1264 | −0,0037 | 0,1589 | −0,0170 | −0,0047 | −0,0003 | −0,5001 | −1,2733 | −0,0003 |

Due to the periodicity of slot placement, for an ideal fluid there is a closed solution to the equation of motion: /149

$$\frac{u}{U}=1+\frac{v_0}{U}\cdot\frac{l}{\pi}\sum_{n=1}^{\infty}\left[a_n'(y)\cos\frac{n\pi x}{l}+b_n'(y)\sin\frac{n\pi x}{l}\right];$$

$$\frac{v}{U}=-n\frac{v_0}{U}\sum_{n=1}^{\infty}\left[a_n(y)\sin\frac{n\pi x}{l}+b_n(y)\cos\frac{n\pi x}{l}\right], \qquad (5.35)$$

where

$$a_n=A_n e^{-\frac{n\pi y}{l}};\quad b_n=B_n e^{-\frac{n\pi y}{l}}$$

To allow for the influence of friction on the wall we will use the Navier-Stokes equations. After introducing the flow function, we can write these equations in the following form:

$$\frac{\partial\psi}{\partial y}\cdot\frac{\partial\Delta\psi}{\partial x}-\frac{\partial\psi}{\partial x}\cdot\frac{\partial\Delta\psi}{\partial y}=\nu\Delta\Delta\psi. \qquad (5.36)$$

We can now expand the flow function into a Fourier series:

$$\psi = Uy + v_\infty \left[ x + \sum_{n=0}^{\infty} \left\{ a_n(y) \cos \frac{n\pi x}{l} + b_n(y) \sin \frac{n\pi x}{l} \right\} \right]. \tag{5.37}$$

Substituting expression (5.37) into equation (5.36), after trans- /150
formations we obtain for each coefficient an ordinary differential equation containing coefficients in the form of products. However, no one has yet succeeded in obtaining a direct solution to this system of nonlinear equations.

An approximate method was proposed to solve this problem. We will take the suction rate to be constant over the entire plate. Then, as we know, an asymptotic velocity profile is established at a certain distance from the leading edge. We will now examine a velocity field as a result of the application on a uniform flow with an asymptotic profile, of a periodically changing supplementary velocity field

$$u = u_0 + u_1; \; v = v_\infty + v_1; \; w = w_0 + w_1, \tag{5.38}$$

where

$$\frac{u_0}{U} = 1 - e^{\frac{v_\infty \eta}{\nu}}; \quad w_0 = \frac{v_\infty}{\nu} e^{\frac{v_\infty y}{\nu}};$$

$u_1$, $v_1$, $w_1$ are periodic functions.

As an initial equation we will take the Navier-Stokes equation in the form

$$u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = \nu \Delta \omega, \tag{5.39}$$

where

$$2\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Taking expression (5.38) into account, we can transform equation (5.39) to the form

$$(u_0 + \underline{u_1}) \frac{\partial \omega_1}{\partial x} + v \left( \frac{\partial \omega_0}{\partial y} + \underline{\frac{\partial \omega_1}{\partial y}} \right) + v_\infty \frac{\partial \omega_1}{\partial y} = \nu \Delta \omega_1. \tag{5.40}$$

If we eliminate the underlined terms of equation (5.40), it becomes linear. However, this elimination is valid only with very low suction rates. The approximate solution obtained can be used as the zero-th approximation with the solution of the complete equation by the iteration method. The differential equation (5.40) corresponds to the "disturbance equation" examined by Pretsch [96] for an asymptotic profile with the difference that in equation (5.40) the boundary conditions for the component $v$ have a somewhat different form.

In reference [130] a calculation was made for a boundary layer which is very thin in comparison to the distance between slots. The obtained results were different from the results for an ideal fluid. /151

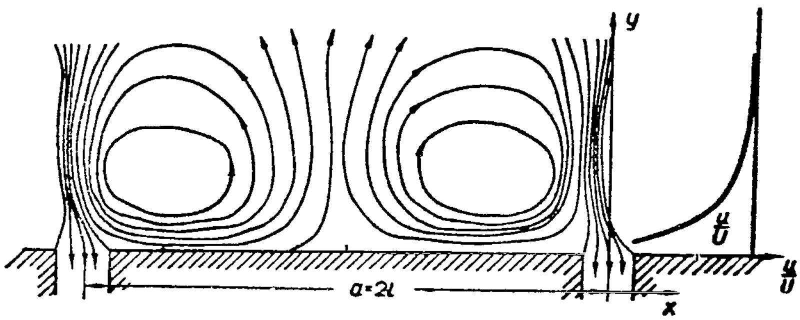



Fig. 49. Streamlines With Slot Suction.

The most significant role in the obtained solution is played by the coefficients $a_1$ and $b_1$. The diagram of the streamlines (Fig. 49) allows us to detect the presence of two eddies. Consequently, midway between the slots the fluid flows from the boundary layer to the external flow.

The presence of eddies can be determined on the basis of equations. We will limit ourselves to the term of equation (5.40)

$$\frac{\partial \omega_1}{\partial x} \approx \frac{1}{u_0} \cdot \frac{d^2 u_0}{dy^2} v_1. \tag{5.41}$$

The obtained diagram of the distribution of eddy intensity with a negative second derivative for the basic profile is shown schematically on Figure 50. It is clear from the figure that the situation is important when the periodic flow has a velocity profile derivative which is non-zero along the normal to the wall. If the thickness of the boundary layer is greater than the distance between the slots, then the disruptions of uniform flow are already damped in the region in front of the wall, in which the velocity profile can be approximated by a straight line. In this case, eddy formation is totally excluded.

Lateral-periodic suction of a boundary layer is of theoretical interest. The potential of the non-viscous flow in the plane $Oxz$ (Fig. 51) in this case can be expressed as /152

$$\varphi = \frac{2v_\infty l}{\pi} \ln \sin \frac{\pi\zeta}{2l}, \qquad (5.42)$$

where

$$\zeta = z + iy.$$

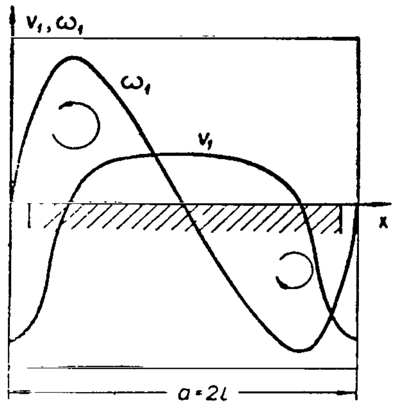



Fig. 50. A Schematic Diagram of the Flow.

The velocity components are

$$v = v_\infty \frac{\sin \frac{\pi z}{l}}{\cos \frac{\pi y}{l} - \cos \frac{\pi z}{l}};$$

$$\omega = v_\infty \frac{\sin \frac{\pi y}{l}}{\cos \frac{\pi y}{l} - \cos \frac{\pi z}{l}}. \qquad (5.43)$$

It is clear from Figure 51 that midway between the slots is the critical point, in the immediate vicinity of which from equation (5.43)

$$v \approx v_\infty \frac{\pi}{2l}(z - z_{cr})$$

$$\omega \approx v_\infty \frac{\pi}{2l} y. \qquad (5.44)$$

These expressions correspond to the flow near the critical point. The action of viscosity on the critical point can be determined by a rigorous solution to the Navier-Stokes equations.

For the viscosity not to exert an influence on the flow near the critical point, the following condition must be satisfied

$$\frac{2v_\infty l}{\nu} \geqslant 100. \qquad (5.45)$$

The Navier-Stokes equations for lateral-periodic uniform suction on a flat plate are significantly simplified, while we obtain for the component $u$ a linear differential equation, in which $v$ and $w$ are known functions in $u$. After $v$ and $w$ are found near the critical point from expressions (5.44), the profile of $u_{cr}$ is determined from the above mentioned differential equation. This the longitudinal velocity profiles near the critical line become known, the profile $u_{cr}$ corresponds to a greater thickness of the boundary layer than the asymptotic profile on the discharge lines (Fig. 52). /153

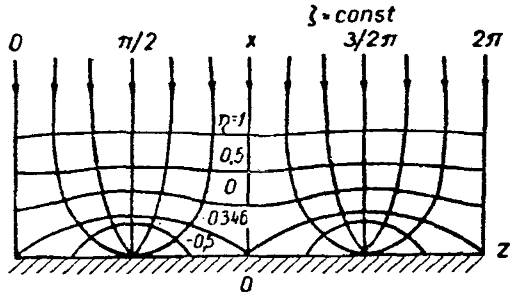



Fig. 51. Curves of the Potential Flow with Slot Suction.

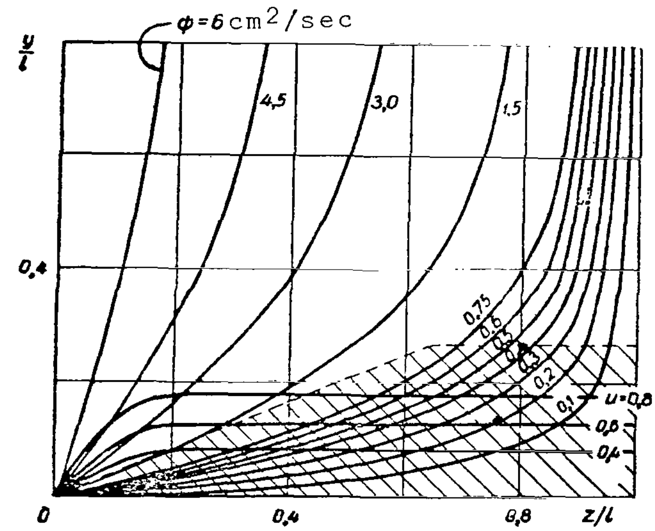



Fig. 52. Streamlines $\psi$ in a Boundary Layer in the Presence of Periodic Suction ($Re = \frac{2lU_0}{\nu} = 100$). The Region of the Boundary Layer is Shaded.

It is interesting to investigate the stability of such a lateral-periodic flow in a laminar boundary layer, when in the cross section perpendicular to the direction of flow, a stable asymptotic velocity profile is combined with large values of the coefficient of friction and the profile of $u_{cr}$ is combined with insignificant stability and small values of the coefficient of friction. /154

## A Simple Approximate Method for Calculating Slot Suction of a Boundary Layer

In the references of Lachmann and Colemann, momentum losses during the flow of the fluid in the region of the slot are not taken into account. It is possible to allow for them by using an equation of three moments [13].

We will examine now uniform suction of a boundary layer on a plate. In this case the equation of the zero-th moment is

$$\frac{d\delta}{dx}+\frac{v_0}{U}=\frac{\tau_0}{\varrho U^2}. \qquad (5.46)$$

Taking expression (3.57) into account we can convert equation (5.46) to the form

$$-\frac{\delta^{**}d\delta^{**}}{\frac{\zeta_0\nu}{U}+(1-d)\frac{-v_0}{U}\delta^{**}}=dx \qquad (5.47)$$

and satisfy the boundary conditions:

$$\text{with } x=x_1 \quad \delta^{**}=\delta_1^{**},$$
$$\text{with } x=x_2 \quad \delta^{**}=\delta_2^{**}.$$

Integrating equation (5.47) after simple algebraic transformations we obtain

$$\frac{(\delta_2^{**}-\delta_1^{**})}{(1-d)-\frac{v_0}{U}}-\frac{\zeta_0\nu}{(1-d)^2\left(\frac{-v_0}{U}\right)^2}\ln\left|\frac{1+\frac{(1-d)}{\zeta_0}\cdot\frac{-v_0}{U}\delta_2^{**}}{1+\frac{(1-d)}{\zeta_0}\cdot\frac{-v_0}{U}\delta_1^{**}}\right|=x_2-x_1. \qquad (5.48)$$

Later we will compute the basic function connecting the characteristics of a laminar boundary layer with the discharge $Q = v_0 s$ with suction of a fluid through a slot of width $s = x_2 - x_1$, located at a distance $x_1$ from the leading edge of the plate. After simple transformations of equation (5.48) we obtain the unknown function /155

$$\frac{Q}{U\delta_1^{**}}=\frac{1}{(1-d)}\left(\frac{\delta_2^{**}}{\delta_1^{**}}-1\right)-\frac{\zeta_0}{(1-d)^2 t_1^{**}}\ln\left|\frac{1+\frac{(1-d)}{\zeta_0}t_1^{**}\frac{\delta_2^{**}}{\delta_1^{**}}}{1+\frac{(1-d)}{\zeta_0}t_1^{**}}\right|, \qquad (5.49)$$

where

$$t_1^{**}=\frac{v_0\delta_1^{**}}{\nu}.$$

It follows from an analysis of formula (5.49) that the dimensionless value of the required discharge varies not only with the ratio of the thicknesses of impulse loss in front of the slot and behind it $\frac{\delta_2^{**}}{\delta_1^{**}}$, as we assumed earlier, but also with the suction parameter $t_1^{**}$ which is the Reynolds number, defined by the thickness of the impulse loss in front of the slot and the local suction rate. The dimensionless discharge as a function of the two parameters is shown in Figure 53.

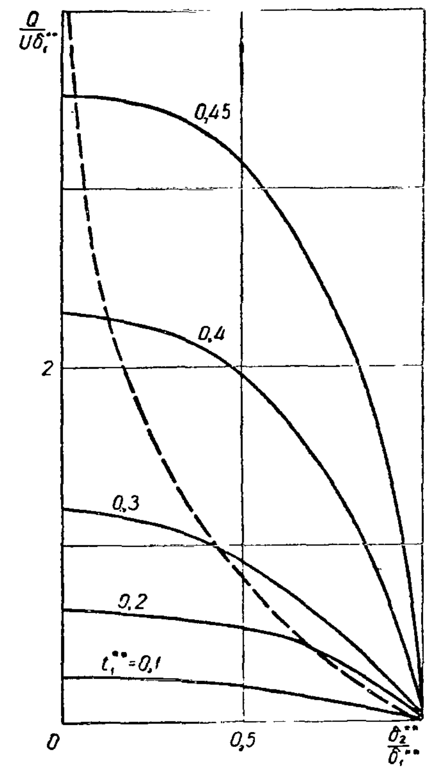



Fig. 53. The Coefficient of Discharge as a Function of the Ratio of the Thicknesses of Impulse Loss in Front of and Behind the Slot: ——— According to the Data of the Author; ······· According to the Data of Lachmann.

Approximate calculations of the characteristics of a laminar boundary layer in the presence of fluid suction across a lateral slot, located on the /156 surface of the plate, should be made using the functions shown in Figure 53 taking the values of the suction paramer into account. The recommended functions can be applied approximately in calculating the characteristics of a laminar boundary layer with slot suction of a fluid from the boundary layer of wing profiles and bodies of revolution and ratios of their length to maximum width greater than seven [29].

Reference [87] is devoted to an experimental investigation of slot suction of a laminar boundary layer. Practical recommendations according to the determination of suction of a laminar boundary layer and hydrodynamic resistence are also made in reference [52].

# CHAPTER 6

# OPTIMAL SUCTION OF A LAMINAR BOUNDARY LAYER /157

## An Approximate Determination of the Lowest Values of the Critical Reynolds Numbers

In the preceding chapters we have examined the problem of the development of a laminar boundary layer with prescribed distribution of the suction rate along the permeable surface of a body. A very important problem in practice also is the decrease in friction resistance due to retention of the laminar condition of flow in the boundary layer by means of suction. It is obvious that the friction resistance, taking into account the power necessary for suction of the fluid from the boundary layer, will be at a minimum, if we choose the local suction rate such that it will be at a minimum and the flow in the laminar boundary layer will continue to be stable. Henceforth we will consider the optimal suction of fluid from the boundary layer across a porous surface to be that distribution of the normal velocity component along the surface, with which in each cross section of the boundary layer the local Reynolds number ($R^{**}$) is equal to its lowest critical value.

To determine the lowest critical Reynolds number we must investigate the hydrodynamic stability of the fluid flow in the boundary layer. These investigations are usually made by the small perturbation method [40, 56]. However, this method for determining the lowest value of the critical Reynolds number is very unwieldy. Therefore, approximate methods for determining the lowest value of the critical Reynolds number have been widely used.

We assume that the velocity profiles in a laminar boundary layer can be described with the required precision by a one-parameter set of curves. In this case the velocity profile in any cross section of the boundary layer can be completely determined by the thickness of the layer and the shape factor. As a shape factor we take the /158
ratio of the conventional thicknesses of the boundary layer

$$H = \frac{\delta^*}{\delta^{**}}, \qquad (6.1)$$

where $\delta^*$ is the displacement thickness and $\delta^{**}$ is the thickness of the impulse loss.

In this case the dimensionless thickness of the boundary layer is characterized fully by the local Reynolds number

$$R^{**} = \frac{U\delta^{**}}{\nu}. \qquad (6.2)$$

An analysis of the results of calculations of the loss of hydrodynamic stability of the flow in the boundary layer showed that the influence of various factors (distribution of the suction rate and velocities in the outer boundary of the layer) on the critical Reynolds number are related to the shape factor $H$. Approximate functions, obtained as a result of calculations for various classes of profiles, are shown in Figure 54. The black dots refer to Hartree profiles and are the results of the calculations for various classes of profiles, are shown in Figure 54. The black dots refer to Hartree profiles and are the results of the calculations of Pretsch [91], [92], [93]. For example, for a velocity profile near the critical point of a plane boundary layer $H$ = 2.22. The corresponding value for an axisymmetric layer is $H$ = 2.33, and for a plate the value is $H$ = 2.59. For a boundary layer on the nonpermeable surface, these points correspond with practically admissible accuracy to the approximate function

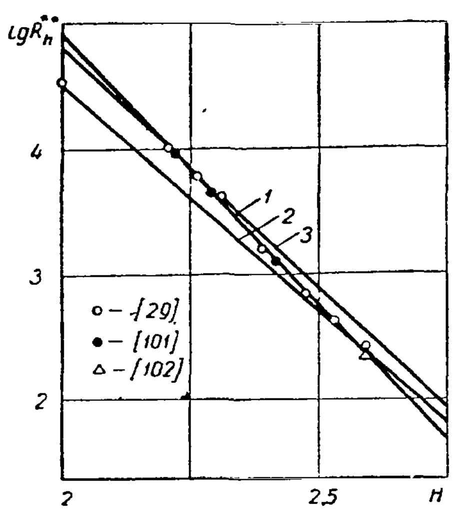



Fig. 54. The Lowest Value of the Criticaly Reynolds Number $R_H^{**}$ as a Function of the Parameter $H$ for a Plate in the Presence of Suction: 1. $R_H^{**}$ = exp (31.3-10$H$); 2. $R_H^{**}$ = exp (26.2-8$H$); 3. $R_H^{**}$ = exp (29.1-9$H$).

$$R^{**} = \exp(A - BH), \qquad (6.3)$$

where $A$ = 26.3 and $B$ = 8 (constant values).

The value of the critical Reynolds number for an asymptotic boundary layer ($H$ = 2) also satisfies the last function. /159

The velocity sets, approximated by sixth degree polynomials (the white dots in Figure 54) correspond to higher values of the critical Reynolds numbers, determined by the same function with $A$ = 29.1 and $B$ = 9.

In reference [123] the hydrodynamic stability of the Schlichting profiles was computed in the presence of suction of a boundary layer. The results obtained are shown in Figure 55 as solid lines.

In the interval of values from $2.2 \lessgtr H \lessgtr 2.6$, which has basic practical value, function (6.3) with $A = 29.1$ and $B = 9$, reflects with sufficient approximation the data of the lowest values of the critical Reynolds numbers for the Schlichting profiles.

Consequently, with the first pair of constants $A = 26.3$ and $B = 8$, formula (6.3) satisfactorily approximates the results of calculations for the stability of the boundary layer of wedges, and also of an asymptotic boundary layer on a porous plate (for $H = 2 - 2.8$). The data of calculations according to formula (6.3) with the second pair ($A = 29.1$ and $B = 9$) of constants show a better correspondence to the lowest critical Reynolds numbers for the Schlichting profiles with suction of a boundary layer and for the six-term Pohlhausen polynomial without suction of the layer (for $H = 2.2 - 2.7$).

The results of subsequent calculations depend to a significant degree on the assigned values of the constants $A$ and $B$. In connection with this fact, new analyses were made of the data of calculation for the hydrodynamic stability of a fluid flow in a laminar boundary layer on a porous plate with uniform suction $v_0 \approx 1/\sqrt{x}$ [102]. This analysis allowed us to establish that in the interval $H = 2.15 - 2.80$ the above-mentioned data can be approximated with a high degree of accuracy (up to 2%) by formula (6.3) for $A = 31.3$ and $B = 10$. Calculations showed that with optimal suction of a laminar boundary layer on a porous plate, this interval of values of $H$ roughly corresponds to a range of Reynolds numbers from $10^5$ to $10^9$.

Figure 54 also shows the data (curves 2 and 3), computed by formula (6.3) for the two pairs of constants ($A = 26.2$; $B = 8$ and $A = 29.1$; $B = 9$). It follows from the data shown in these graphs, that with $A = 31.3$ and $B = 10$, formula (6.3) shows a better correlation than the above-mentioned data with the results of calculations for the lowest critical Reynolds number for a porous plate with suction of the boundary layer.

Figure 55 shows the lowest critical Reynolds number, for different pressure gradients and suction parameters, as a function of the shape factor of the boundary layer for the Schlichting, Hartree and six-term polynomial profiles. These data can be used for an approximate evaluation of the lowest critical Reynolds number. /160

### Simultaneous Use of the Impulse and Energy Relationships

Near the leading critical point the boundary layer remains laminar without suction. At the point of stability loss a danger of turbulence arises. Beginning at this point for a nonpermeable surface it follows to have optimal suction of the boundary layer, satisfying relationship (6.3). To calculate the optimal suction of a laminar boundary layer in addition to relationship (6.3) K. Wieghardt [123] proposed the use of the integral impulse and energy relationships

$$\frac{1}{r_0}\cdot\frac{d}{dx}(r_0\delta^{**}) +$$ /161

$$+(2+H)\frac{\delta^{**}}{U}\cdot\frac{dU}{dx} - \frac{v_0}{U} = \frac{\tau_0}{\varrho U^2} = \frac{\varepsilon}{R^{**}};$$

$$\frac{1}{U^3 r_0}\cdot\frac{d}{dx}(U^3 r_0\delta^{***}) - \frac{v_0}{U} = \frac{2D}{R^{**}}, \qquad (6.4)$$

where

$$\varepsilon = \frac{\partial(u/U)}{\partial(y/\delta^{**})},$$

$$D = \int_0^{\delta}\frac{r}{r_0}\left[\frac{\partial(u/U)}{\partial(y/\delta^{**})}\right]^2 d\left(\frac{y}{\delta^{**}}\right).$$

Relationship (6.4) is valid for a laminar boundary layer with axial symmetry. In order to obtain the relationships for a plane boundary layer, we should take in system (6.4) the value $r_0 \approx 1$.

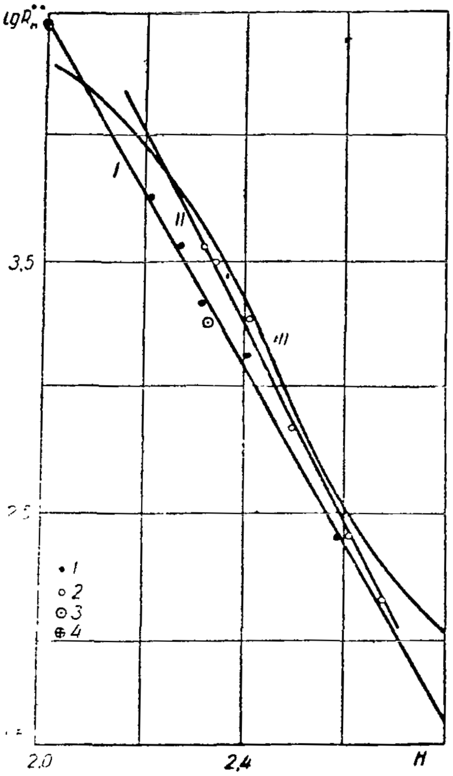



Fig. 55. The Lowest Critical Reynolds Number for Different Pressure Gradients and Suction Parameters as a Function of the Shape Factor of the Boundary Layer: (1) The Hartree Profiles; (2) The Six-Term Polynomial [101]; (3) The Profile at the Critical Point; (4) An Asymptotic Profile with Suction; (*I*) $R^{**}_H$ = exp (29.3-8*H*); (*II*) $R^{**}_H$ = exp(29.1-9*H*); (*III*) The Schlichting Profiles [103] in the Presence of Suction.

For one-parameter profile sets in the boundary layer there are single-valued relationships between the parameters $\varepsilon$, $D$ and $H$. Using these functions we obtain from system (6.4) differential equations with respect to the values of $H$ and $R^{**}$.

$$\frac{dH}{d\bar{s}} = -f(H)\frac{1}{U/U_0}\cdot\frac{d(U/U_0)}{d\bar{s}} - g(H)\frac{U}{U_0}\cdot\frac{\mathrm{Re}}{R^{**2}} + h(H)\frac{\mathrm{Re}}{R^{**}}\cdot\frac{v_0}{U_0};$$

$$\frac{dR^{**}}{d\bar{s}} = -\left\{\frac{H+1}{(U/U_0)}\cdot\frac{d(U/U_0)}{d\bar{s}} + \frac{1}{r_0/L}\cdot\frac{d(r_0/L)}{d\bar{s}}\right\}R^{**} + \varepsilon\frac{U}{U_0}\cdot\frac{\mathrm{Re}}{R^{**}} + \mathrm{Re}\frac{v_0}{U_0}. \qquad (6.5)$$

Here $L$ is the length of the wing profile or body of revolution; $\bar{s} = \frac{x}{L}$ is the dimensionless length of the arc measured from the leading critical point; $U_0$ is the velocity of the leading flow; $\mathrm{Re} = \frac{U_0 L}{\nu}$ is the Reynolds number; $f$, $g$ and $h$ are functions of the parameter $H$ for which the following formulas are valid: /162

$$\left.\begin{aligned} f(H) &= -\frac{\delta^{***}}{\delta^{**}} \cdot \frac{(H-1)}{N}; \\ g(H) &= -\frac{2D - \varepsilon \frac{\delta^{***}}{\delta^{**}}}{N}; \\ h(H) &= -\frac{\frac{\delta^{***}}{\delta^{**}} - 1}{N} \end{aligned}\right\} \tag{6.6}$$

where

$$N = \frac{d\,\frac{\delta^{***}}{\delta^{**}}}{dH}.$$

The values of the functions $f$, $g$, $h$ and their derivatives are shown in Table 12.

TABLE 12. VALUES OF THE AUXILIARY FUNCTIONS.[1]

| $H$ | $f$ | $g$ | $h$ | $\frac{1}{2}\frac{df}{dH}$ | $\frac{1}{2}\frac{dg}{dH}$ | $\frac{1}{2}\frac{dh}{dH}$ | $\varepsilon$ |
|---|---|---|---|---|---|---|---|
| 2,00 | 8,0 | 1,60 | 3,19 | 7,0 | 1,15 | 0,7 | 0,500 |
| 2,05 | 8,7 | 1,48 | 3,27 | 7,4 | 1,10 | 0,9 | 0,467 |
| 2,10 | 9,5 | 1,37 | 3,37 | 8,1 | 1,05 | 1,1 | 0,437 |
| 2,15 | 10,3 | 1,27 | 3,49 | 8,9 | 1,05 | 1,3 | 0,409 |
| 2,20 | 11,3 | 1,17 | 3,63 | 10,0 | 1,05 | 1,5 | 0,383 |
| 2,25 | 12,4 | 1,06 | 3,79 | 11,1 | 1,10 | 1,65 | 0,359 |
| 2,30 | 13,6 | 0,95 | 3,96 | 12,2 | 1,10 | 1,75 | 0,337 |
| 2,35 | 14,9 | 0,84 | 4,14 | 13,4 | 1,10 | 1,85 | 0,316 |
| 2,40 | 16,3 | 0,73 | 4,33 | 14,7 | 1,15 | 2,0 | 0,297 |
| 2,45 | 17,8 | 0,61 | 4,54 | 16,2 | 1,25 | 2,25 | 0,279 |
| 2,50 | 19,5 | 0,48 | 4,78 | 18,3 | 1,35 | 2,6 | 0,262 |
| 2,55 | 21,5 | 0,34 | 5,06 | 20,6 | 1,40 | 3,0 | 0,246 |
| 2,60 | 23,7 | 0,20 | 5,38 | 23,0 | 1,45 | 3,35 | 0,231 |
| 2,65 | 26,1 | 0,06 | 5,73 | 25,5 | 1,50 | 3,65 | 0,217 |
| 2,70 | 28,7 | 0,10 | 6,11 | 28,1 | 1,60 | 3,95 | 0,204 |
| 2,75 | 31,7 | 0,27 | 6,52 | 31,0 | 1,75 | 4,25 | 0,192 |
| 2,80 | 34,9 | 0,45 | 6,96 | 34,6 | 2,00 | 4,6 | 0,181 |
| 2,85 | 38,7 | 0,67 | 7,44 | 40,0 | 2,40 | 4,9 | 0,170 |

[1] In compiling this table, we used as a basis the Schlichting profiles with suction.

Using expression (6.3) and the system of equations (6.4) we obtain /163

$$\frac{dH}{d\bar{s}} = -\frac{f_1(H)}{U} \cdot \frac{dU}{d\bar{s}} + \frac{f_2(H)}{r_*} \cdot \frac{dr_*}{d\bar{s}} - f_3(H)\frac{U}{U_0} \cdot \frac{\mathrm{Re}}{R^{**}}; \tag{6.7}$$

$$R^{**} = \exp(A - BH).$$

Linearizing the first equation of system (6.7) for a small increment $\Delta\bar{s}$, we obtain

$$\Delta H = \frac{-f_1\gamma + f_2\omega - f_3\frac{\mathrm{Re}\bar{U}}{R^{**2}U_0}\Delta\bar{s}}{1 + \frac{1}{2}\dot{f}_1\gamma + \frac{1}{2}\dot{f}_2\omega + \frac{1}{2}(\dot{f}_3 + 2Bf_3)\frac{\mathrm{Re}}{R^{**}}\frac{\bar{U}\Delta\bar{s}}{U_0}}, \tag{6.8}$$

where

$$\bar{U} = \frac{1}{2}(U_1 + U_2); \quad \omega = \frac{1}{r_0}\cdot\frac{dr_0}{ds}\Delta\bar{s} \approx \frac{2\left[(r_0)_2 - (r_0)_1\right]}{(r_0)_2 + (r_0)_1};$$

$$\gamma \approx \frac{1}{U}\cdot\frac{dU}{ds}\Delta\bar{s} \approx \frac{2\left[U_2 - U_1\right]}{U_2 + U_1}; \quad \dot{f}_1 = \frac{df_1}{dH}; \quad \dot{f}_2 = \frac{df_2}{dH}; \quad \dot{f}_3 = \frac{df_3}{dH}$$

After computing the values of $H(\bar{s})$ and, consequently, also $R^{**}$, we can determine from the integral impulse relationship the local suction rate

$$\frac{v_0}{U_0} = \frac{1}{\mathrm{Re}}\cdot\frac{dR^{**}}{ds} + \left(\frac{H+1}{U}\cdot\frac{dU}{ds} + \frac{1}{r_0}\cdot\frac{dr_0}{ds}\right)\frac{R^{**}}{\mathrm{Re}} - \frac{U\varepsilon}{U_0R^{**}}. \tag{6.9}$$

The auxiliary functions $f_1$, $f_2$ and $f_3$ in this case vary with $H$ and are approximated by the approximate formulas

$$\left.\begin{aligned} f_1(H) &= -\frac{H + 1 - 2H_{32}}{N_1} \approx -0{,}0121 + 0{,}2745\,(H - 2{,}2); \\ f_2(H) &= \frac{1 - H_{32}}{N_1} \approx 0{,}1210 + 0{,}0033\,(H - 2{,}2); \\ f_3(H) &= \frac{\varepsilon - 2D}{N_1} \approx 0{,}0385\,(H - 2), \end{aligned}\right\} \tag{6.10}$$

where

$$N_1 = B(1 - H_{32}) + \frac{dH_{32}}{dH};$$

$$\overset{*}{H}_{32} = \frac{\overset{*}{\delta}{}^{***}}{\delta^{**}}.$$

Thus all formulas have been obtained to calculate the optimal suction of a boundary layer. /164

Figure 56 shows the curves obtained as a result of calculations of the optimal suction of a boundary layer for an ellipsoid of rotation with a semi-axis ratio 1:8 with Re = $8 \cdot 10^6$. From the initial point of suction for approximately 75% of the ellipse the suction rate remains practically constant ($\frac{v_0}{U_0} = 1.4 \cdot 10^{-4}$).

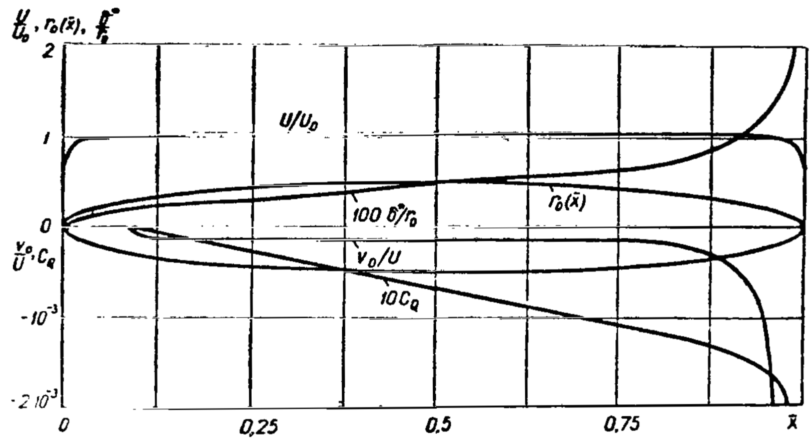



Fig. 56. Optimal Distribution of the Suction Rate for an Ellipsoid of Rotation (According to the Data of Wieghardt).

The Wieghardt method for calculating the optimal suction of a laminar boundary layer was in practice very unwieldy and time-consuming. Its deficiencies were basically related to the use of the method of finite differences to obtain numerical results. To eliminate these shortcomings, Wortmann [125] proposed instead of numerical integration a more simple method of solution.

Using equation (6.3) we can transform the second equation of system (6.5) to the form /165

$$\frac{dR^{**}}{d\bar{s}} = \left(A_1 \frac{dU}{d\bar{s}} \cdot \frac{1}{U} + A_2 \frac{dr_0}{d\bar{s}} \cdot \frac{1}{r_0}\right) R^{**} + A_3 \frac{U}{U_0} \cdot \frac{\mathrm{Re}}{R^{**}}, \quad (6.11)$$

where

$$A_1 = \frac{f - f_1}{h} - (H + 1); \qquad A_2 = 1 + \frac{f_2}{h}; \qquad A_3 = \frac{g - f_3}{h} + \varepsilon,$$

and the functions $f_1$, $f_2$, $f_3$ vary, as do $f$, $g$, $h$ only with $H$ and have the previous values.

Taking $H$ = const., we integrate equation (6.11) by the same method we used to solve the equation for a boundary layer without suction, and we obtain

$$R^{**2}=R_H^{**2}+2\,\mathrm{Re}\,A_8\left(\frac{U}{U_0}\right)^{2A_1}\left(\frac{r_0}{L}\right)^{2A_2}\int_{\bar{s}_n}^{\bar{s}}\left(\frac{U}{U_0}\right)^{1-2A_1}\left(\frac{r_0}{L}\right)^{-2A_2}d\bar{s}. \quad (6.12)$$

Calculations begin from a value of the Reynolds number $R_H^{**}$ at the point of loss of layer stability with respect to small perturbations. For the given values of $U(\bar{s})$ and $r_0(\bar{s})$ and various shape factors, we can compute $R^{**}$. If we plot the set of curves of $R^{**}(s, H_n)$ and the critical Reynolds numbers corresponding to each value of $H$, then we can determine graphically the unknown function $R^{**}_{cr}(\bar{s})$. When the function $R^{**}_{cr}(\bar{s})$ is known, we can compute the local suction rate from the expression

$$\frac{v_0}{U_0}=\frac{f-f_1}{h}\cdot\frac{dU}{ds}\cdot\frac{1}{U}\cdot\frac{R^*}{\mathrm{Re}}+\frac{(g-k_3)}{h}\cdot\frac{U}{U_0R^{**}_{cr}}. \quad (6.13)$$

A further simplification of the Wieghardt-Wortmann method is presented in reference [132].

## The Use of Equations of "Three Moments"

The problem of optimal suction of fluid from the boundary layer of a porous plate was first solved in reference [91] by numerical integration of the Prandtl equation. References [16, 123, 125] are also devoted to an approximate solution to the analogous problem of a boundary layer with longitudinal pressure drop on the outer boundary. In these calculations, the impulse and energy equations were used. We will now find a system of equations of the zero-th and second moments in order to calculate the optimal suction of fluid from a boundary layer [18]. /166

We can write the equation of the zero-th moment in the form

$$\frac{df}{dx}=\frac{U''}{U'}f+\frac{U'}{U}[a+(B-2)t^{**}-bf], \quad (6.14)$$

where $a$ = 0.44; $b$ = 5.48 and $B$ = 1.12 (constant values).

Since

$$f=\frac{\delta^{**2}U'}{\nu}=\frac{\nu U'}{U^2}R^{**2} \qquad \left(R^{**}=\frac{U\delta^{**}}{\nu}\right), \quad (6.15)$$

differentiating equation (6.15), we find

$$\frac{df}{dx}=\frac{\nu U'}{U^2}R^{**2}-\frac{2\nu U'^2}{U^3}R^{**2}+\frac{\nu U'}{U^2}\cdot\frac{dR^{**2}}{dx}. \tag{6.16}$$

Substituting formulas (6.15) - (6.16) into equation (6.14) and making algebraic transformations, we obtain

$$\frac{\nu}{U}\cdot\frac{dR^{**2}}{dx}+\frac{\nu U'}{U^2}(b-2)R^{**2}-(B-2)\frac{v_0}{U}R^{**}-a=0$$

$$\left(t^{**}=\frac{v_0}{U}R^{**}\right), \tag{6.17}$$

where $v_0(x)$ is the local suction rate from the boundary layer.

We can write the equation of the second moment in the form

$$\frac{df}{dx}=\frac{U''}{U'}f+\frac{U'}{U}\cdot\frac{a}{H_0}(H-H_4t^{**}-H_0c\bar{f}), \tag{6.18}$$

where $H$ is the shape factor of the boundary layer, and $c$ = 9.54; $H_0$ = 2.59 and $H_4$ = 4 (constant values).

After substituting formulas (6.15) - (6.16) into equation (6.18) and making the corresponding algebraic transformations we obtain

$$\frac{\nu}{U}\frac{dR^{**2}}{dx}+\frac{\nu U'}{U^2}(ac-2)R^{**2}+a\frac{H_4}{H_0}\frac{v_0}{U}R^{**}-a\frac{H}{H_0}=0. \tag{6.19}$$

Excluding the value $v_0/U$ from equations (6.17) and (6.19), we obtain the differential equation for computing the local Reynolds /167
number

$$\frac{dR^{**2}}{dx}+\frac{U'}{U}k_0R^{**2}-\frac{\frac{a}{H_0}\left(\frac{aH_4}{B-2}-H\right)}{\frac{\nu}{U}\left(1+\frac{a}{B-2}\cdot\frac{H_4}{H_0}\right)}=0, \tag{6.20}$$

where

$$k_0=\frac{\left[(ac-2)+\frac{a(b-2)}{B-2}\cdot\frac{H_4}{H_0}\right]}{\left(1+\frac{a}{B-2}\cdot\frac{H_4}{H_0}\right)}.$$

Integrating equation (6.20), satisfying the boundary condition $R^{**} = R_0^{**}$ when $x = x_0$, where $x_0$ is the coordinate of the point of stability loss without fluid suction, we find

$$R^{**2} = \frac{a}{(B-2)H_0 + aH_4} \frac{[U(x)]^{-k_0}}{\nu} \int_{x_0}^{x} [aH_4 - H(B-2)] U^{k_0+1}(x)\,dx + R_0^{**2}\left[\frac{U(x_0)}{U(x)}\right]^{k_0}. \tag{6.21}$$

The calculations of the optimal suction of fluid from a boundary layer begin from the value of the local Reynolds number $R_0^{**}$ at the point of loss of layer stability, before which the flow without fluid suction in the layer is stable with respect to small perturbations. We can compute $R_0^{**}$ according to formula (6.21) for the known velocity distriubtion on the outer boundary of the layer of the body and the given values of the shape factor $H$. After obtaining the set of curves for $R^{**}(x,H)$ and the local critical Reynolds numbers corresponding to each value of the shape factor $H$, we can graphically determine the function $R_{cr}^{**}(x)$. In this case the values of $R_{cr}^{**}(H)$ should be computed by formula (6.3).

We obtain from the equation of the zero-th moment (6.17) a formula for computing the optimal distribution of the rate of fluid suction

$$\frac{v_0}{U} = \frac{\nu}{U}\cdot\frac{1}{B-2}\cdot\frac{1}{R^{**}}\cdot\frac{dR^{**2}}{dx} + \frac{\nu U'(b-2)}{U^2(B-2)}R^{**} - \frac{a}{B-2}\cdot\frac{1}{R^{**}}. \tag{6.22}$$

We can determine the first term in formula (6.22) from differential equation (6.20). After the corresponding transformations we obtain /168

$$\frac{\nu}{U}\cdot\frac{1}{B-2}\cdot\frac{1}{R^{**}}\cdot\frac{dR^{**2}}{dx} = -\frac{\nu U'}{U^2}\cdot\frac{k_0}{B-2}R^{**} + \frac{a}{(B-2)}\cdot\frac{[aH_4 + H(B-2)]}{[H_0(B-2) + aH_4]}\cdot\frac{1}{R^{**}}. \tag{6.23}$$

Substituting this expression into equation (6.22) and making the necessary calculations we obtain in final form the formula for determining the optimal distribution of the rate of suction of fluid from the boundary layer:

$$\frac{v_0}{U} = \frac{\nu U'}{U^2}\cdot\frac{(b-2-k_0)}{B-2}R^{**} + \frac{a(H-H_0)}{[H_0(B-2) + aH_4]}\cdot\frac{1}{R^{**}}. \tag{6.24}$$

Having determined the optimal distribution of the suction rate from formula (6.24), we can compute by a known method all the characterisitics of a boundary layer and the friction resistance.

This method for calculating the distribution of the rate of optimal suction of fluid from a boundary layer was developed for plane flows. With the help of the known Stepanov-Mangler transformation this method can also be applied to axisymmetric flows.

## The Influence of Initial Turbulence and Surface Roughness on the Optimal Suction

Above we considered the optimal suction of fluid from the boundary layer to be that distribution of the normal velocity component along the surface of the body when the local Reynolds number in each cross section of the boundary layer is equal to its lowest critical value, computed by the method of small perturbations according to the Tollmien-Schlichting theory of hydrodynamic stability [99, 114]. Analysis of the experimental data showed that laminarization of the boundary layer by suction of fluid across the porous surface of a body is possible with the condition that the local Reynolds number is equal to its critical value at the point of transfer varying with the initial turbulence of the flow and the roughness of the surface of the body. This circumstance has great pracitcal value in calculations of a laminar boundary /169
layer on bodies having a certain roughness of their surface and moving in a medium with low initial turbulence. Therefore, it is very important to determine the possible influence of initial turbulence and surface roughness on the optimal distribution of the rate of suction of a small quantity of fluid across a specially prepared porous surface of a body.

We turn now to the determination of the critical Reynolds number at the transition point. According to the known Taylor hypothesis [109] we will assume from now on that turbulence in a laminar layer arises by action of perturbations of finite amplitude caused by eddies originating at the surface of the body and formed due to premature local separations of the boundary layer. In this case[2] the finite perturbations in the boundary layer are caused by turbulence of the leading flow or by eddies formed with streamlining of the elements of roughness located on the surface of the body [15].

As we know, premature local separation of the laminar layer can be determined approximately by the relationship

$$\frac{\delta_t^{**2}}{\nu}\cdot\frac{1}{\varrho U}\left\{\frac{dp}{dx}-\left[\frac{\widetilde{\partial p}}{\partial x}\right]\right\}=f_r. \qquad (6.25)$$

[2] The influence of initial turbulence and surface roughness are discussed in references [9, 10, 14, 24].

where $\frac{dp}{dx}$ is the longitudinal pressure gradient on the outer boundary of the boundary layer; $\left[\frac{\partial \tilde{p}}{\partial x}\right]$ is the root squared from the mean-square value of pulsation of the longitudinal pressure gradient on the outer boundary of the layer; $\delta_t^{**}$ is the thickness of the impulse loss of the boundary layer at the point of transfer; $U$ is the velocity on the outer boundary of the layer; $f_s$ is the value of the shape factor at the point of separation of the boundary layer.

The value of the shape factor $f_s$[3] varies with the intensity of suction of fluid across the surface of a body, characterized by a suction parameter

$$t^{**} = \frac{v_0 \delta^{**}}{\nu},$$

where $v_0(x)$ is the local rate of suction of the fluid across the surface. Figure 57 shows the shape factor $f_s$ as a function of the suction parameter $t^{**}$. /170

Since pressure pulsations, caused by turbulent motion of the fluid, follow the law of random events, we can assume that the mean-square value of pressure pulsations on the outer boundary of the boundary layer follow the summation rule:

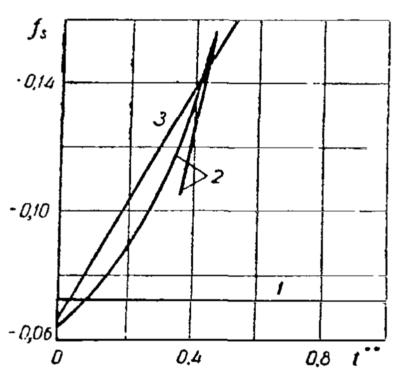



Fig. 57. The Shape of Factor of the Point of Boundary Layer Separation as a function of the Suction Parameter $t^{**}$: (1) According to the Data of [103]; (2) According to the Data of [115]; (3) According to the Data of [117].

$$\left[\frac{\partial \tilde{p}}{\partial x}\right] = \left[\frac{\partial \tilde{p}}{\partial x}\right]_1 + \left[\frac{\partial \tilde{p}}{\partial x}\right]_2, \qquad (6.26)$$

where $\left[\frac{\partial \tilde{p}}{\partial x}\right]_1$ is the component caused by turbulence in the external flow; $\left[\frac{\partial \tilde{p}}{\partial x}\right]_2$ is the component caused by eddies formed with streamlining of an element of roughness located on the surface of the body.

We will assume that the turbulence of the leading flow is isotropic. If the transition point is located at a certain distance from the leading critical point, then the surface of the body does not distort the isotropy of the turbulence on the outer boundary

[3] The question of the transition of a laminar boundary layer to a turbulent one with suction is examined in reference [12].

of the boundary layer. Then for the connection between the pulsations of the longitudinal pressure gradient $\left[\frac{\widetilde{\partial p}}{\partial x}\right]_1$ and the velocities on the outer boundary of the boundary layer we can use the relationships of the statistical theory of turbulence [109]:

$$\left[\frac{\widetilde{\partial p}}{\partial x}\right]_1 \sim \frac{\varrho u_1'^2}{\lambda_1}; \qquad (6.27)$$

$$\lambda_1 \sim L\sqrt{\frac{\nu}{u_1' L}}, \qquad (6.28)$$

where $u_1'$ is the mean-square value of velocity pulsation on the outer boundary of the boundary layer; $\lambda_1$ is the dimension of the "smallest eddy"; $L$ is the turbulence scale in the external flow. /171

For a pulsed flow, caused by eddies separated from an element of roughness[4], located on the surface of the body, we have

$$\left[\frac{\widetilde{\partial p}}{\partial x}\right]_2 \sim \frac{\varrho u_2'^2}{\lambda_2}; \qquad (6.29)$$

$$u_2' \sim U\left(\frac{d}{\delta_d^{**}}\right); \qquad (6.30)$$

$$\lambda_2 \sim d, \qquad (6.31)$$

where $u_2'$ is the mean-square value of the velocity pulsation on the outer boundary of the layer; $\lambda_2$ is the dimension of the "smallest eddy"; $d$ is the height of the element of roughness; $\delta_d^{**}$ is the thickness of the impulse loss at the location of the roughness.

Separating with the help of relationships (6.26) - (6.28) the critical Reynolds number $R_t^{**} = \frac{U\delta_t^{**}}{\nu}$ and the Taylor parameter $\varepsilon\left(\frac{L}{L_0}\right)^{1/5}$ we can transform the transition condition (6.25) to the form

$$R_t^{**} = R_{l/}^{**} + \frac{A(f_s + f)^{\frac{1}{2}}}{\left[\varepsilon\left(\frac{L}{L_0}\right)^{\frac{1}{5}}\right]^{\frac{5}{4}}\left[\left(\frac{u_0'}{u_1'}\right)\left(\frac{U_0}{U}\right)\left(\frac{L_0}{L\delta}\right)^{\frac{1}{5}}\right]^{\frac{5}{4}} + B\left(\frac{Ud}{\nu}\right)^{-\frac{1}{2}}\left(\frac{d}{\delta_d^{**}}\right)^2}, \qquad (6.32)$$

---

[4] In reference [11] it was shown that the universality of the Dryden function [61], which evaluates the influence of surface roughness on the transition of a laminar boundary layer to a turbulent one, is not corroborated by experiments. The question of the internal resistance of an element of roughness is discussed in reference [21].

where $\varepsilon = \frac{u_0'}{U_0}$ is the degree of turbulence in the leading flow; $U_0$ is the velocity of the leading flow; $u_0'$ is the mean-square value of the velocity pulsation in the leading flow and $L$ is the length of the body.

The value $R_H^{**}$ is introduced into equation (6.32) to satisfy the condition of the damping of perturbations in the boundary layer from the turbulence of the external flow and surface roughness, when the critical Reynolds number is equal to its lowest value. It is necessary to do this because when $R_t^{**} \to R_H^{**}$, the pulsed pressure gradient on the outer boundary of the boundary layer due to the rapid damping of the pulsations does not significantly affect the transition. The validity of this transformation is confirmed by an experiment. /172

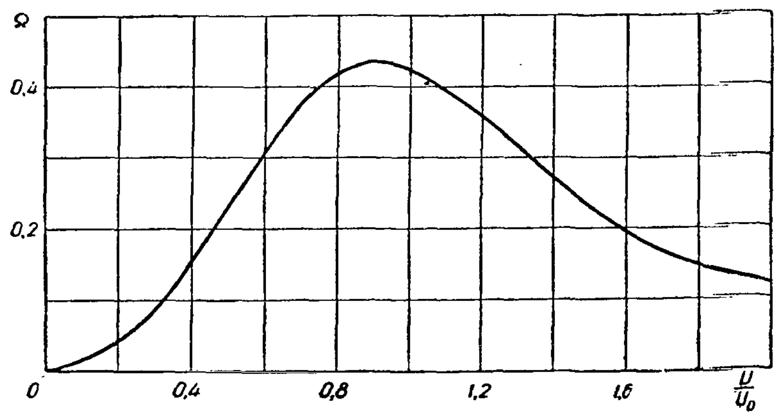



Fig. 58. The Curve of the Values of the Function $\Omega(\frac{U}{U_0})$.

The values of the lowest critical Reynolds numbers for different pressure gradients and laws of fluid suction distribution can be computed by the approximate formula (6.3). For a more precise calculation of the lowest values of the critical Reynolds numbers, we must examine the stability of the fluid flow in the boundary layer.

The values $u_1$ and $L_0$, which were introduced into formula (6.32), are the mean-square value of the velocity pulsation and the turbulence scale on the outer boundary of the layer respectively. Since the turbulence characteristics $u_0'$ and $L$ are usually known in our calculations, it is necessary to express $u_1'$ and $L_\delta$ in terms of the above-mentioned characteristics. To derive this function we will use the known relationships [6]

$$\frac{u_1'}{u_0'} \sim \frac{\left(\frac{U}{U_0}\right)^3}{1+\left(\frac{U}{U_0}\right)^4}; \tag{6.33}$$

$$\frac{L_\delta}{L_0} \sim \left(\frac{U}{U_0}\right). \tag{6.34}$$ /173

Substituting relationships (6.33) and (6.34) into formula (6.28) and making the necessary transformations, we obtain the final expression for the critical Reynolds number

$$R_t^{**} = R_H^{**} + \frac{A(f_s+f)^{\frac{1}{2}}}{\left[\varepsilon\left(\frac{L}{L_0}\right)^{\frac{1}{5}}\right]^{\frac{5}{4}} \mathrm{Re}^{-\frac{1}{4}}\Omega\left(\frac{U}{U_0}\right) + B\left(\frac{Ud}{\nu}\right)^{-\frac{1}{2}}\left(\frac{d}{\delta_d^{**}}\right)^2}. \tag{6.35}$$

Figure 58 shows the curves of the function $\Omega\left(\frac{U}{U_0}\right) = \frac{\left(\frac{U}{U_0}\right)^3}{\left[1+\left(\frac{U}{U_0}\right)^4\right]^{\frac{5}{4}}}$.

If the surface of the body is a smooth one, for the given turbulence of the leading flow the difference $R_t^{**} \to R_H^{**}$ is proportional to the value $(f_s + f)^{1/2}$ of (6.35). At the same time according to the data of reference [45], if we take into account the relationship of the lowest critical Reynolds number and the shape factor $f$, the above-mentioned difference is proportional to the value $(f_s + f)^{2/3}$. Comparison of the results of the calculations with experimental data (Fig. 59) show that formula (6.35) corresponds more precisely to reality.

Equation (6.35) includes the two constant values $A$ and $B$ the values of which could not be determined theoretically. Analysis of experimental data [59, 60] showed that $A = 0.22$ and $B = 0.21$.

To verify the semi-empirical formula (6.35) we compared the results of calculations and experiments [60] on an elliptical cylinder with a semi-axis ratio $\frac{a}{b} = 3$. The calculated and experimental data agree satisfactorily (Fig. 60).

In practical calculations the turbulence scale is not always known, since its determination requires very accurate and time-consuming measurements. Equation (6.35) includes this value to the 1/4 power, so that even with significant changes in its value the critical Reynolds number varies very little. The value of the functions $\Omega$ and $(\frac{Ud}{\nu})^{1/2}$ in calculations of the transition point also vary insignificantly.

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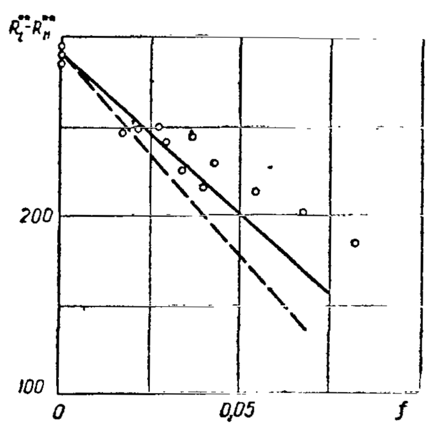


Fig. 59. Comparison of the Computed Function $R_t^{**} - R_H^{**}$ with Experimental Data: ——— According to Formula (6.35); ····· According to the Data of [45]; o Shows the Experimental Data.

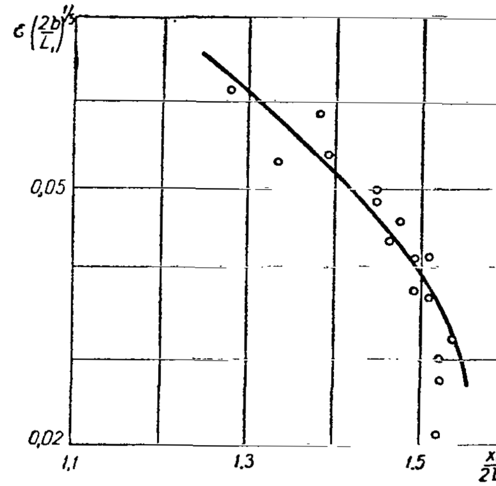


Fig. 60. Comparison of the Results of Calculations of the Transition Point with Experimental Data: ——— the Calculation According to Formula (6.35); o Shows the Experimental Data.

Comparison of the calculated and experimental data shows that /175 we can take

$$A\left(\frac{U_0L}{\nu}\right)^{\frac{1}{4}}\Omega^{-1}=2{,}0; \tag{6.36}$$

$$\frac{A}{B}\left(\frac{Ud}{\nu}\right)=2250. \tag{6.37}$$

Then formula (6.32) is significantly simplified and reduced to the form

$$R_t^{**}=R_H^{**}+\frac{C(f_s+f)^{\frac{1}{2}}}{\varepsilon^{\frac{5}{4}}+D\left(\frac{d}{\delta_d^{**}}\right)^2}, \tag{6.38}$$

which is very convenient for practical use. The constants $C$ and $D$ in formula (6.38) are equal to 1.88 and $0.25\cdot10^{-3}$ respectively.

The critical Reynolds number $R_t^{**}$ for a plate as a function of the degree of turbulence and the surface roughness parameter

$$R^{**}_{t\,\mathrm{p\,I}} = 225 + \frac{0{,}089^{\frac{1}{2}} C}{\varepsilon^{\frac{5}{4}} + D\left(\frac{d}{\delta^{**}_d}\right)^2} \qquad (6.39)$$

are shown in Figure 61.

The calculated and experimental values of the critical Reynolds numbers as a function of the degree of turbulence of the leading flow are shown in Figure 62. We can consider the agreement to be completely satisfactory after we exclude the portion of the curve corresponding to an insignificant (not more than 0.1%) level of turbulence in the leading flow.

Figure 63 shows the calculated and experimental values of the critical Reynolds numbers for a body with a different surface roughness (6.38). Experiments were conducted with low ($\varepsilon$ = 0.15%) and comparatively high ($\varepsilon$ = 0.5-0.6%) levels of turbulence in the leading flow. Comparison shows a satisfactory correlation of results computed according to formula (6.38) and experimental data, except for the case of small values of turbulence and surface roughness ($\varepsilon \lesssim$ 0.15%). The reason for the discrepancy in this case is the unsuitability of the transition scheme used in this discussion for /176
very small velocity perturbations, arising in the boundary layer due to turbulence of the leading flow or roughness of the surface of the body.

It is interesting to note that a comparison of the obtained data with the graphs of calculations of the transition points, presented in references [68, 69, 90] shows a correlation between the above mentioned graphs and formula (6.38) with $\varepsilon$ = 0.15-0.35%. This fact is additional confirmation of the validity of formula (6.38), since the experimental data used to compare these graphs were obtained in the designated interval of the level of turbulence of the leading flow.

Substituting expressions (3.43), (3.47) and (6.3) into formula (6.38), after simple algebraic transformations we obtain the final expression for determining the critical Reynolds number at the transition point [25]:

$$R_t^{**} = \exp(A_1 - B_1 H) + \frac{C_1\left[f_{\varepsilon_0} + \frac{H}{m}\left(\frac{C_2}{m} - k\right)t^{**} - \frac{H}{m}\right]^{\frac{1}{2}}}{\varepsilon^{\frac{5}{4}} + D\left(\frac{d}{\delta^{**}_d}\right)^2} \qquad (6.40)$$

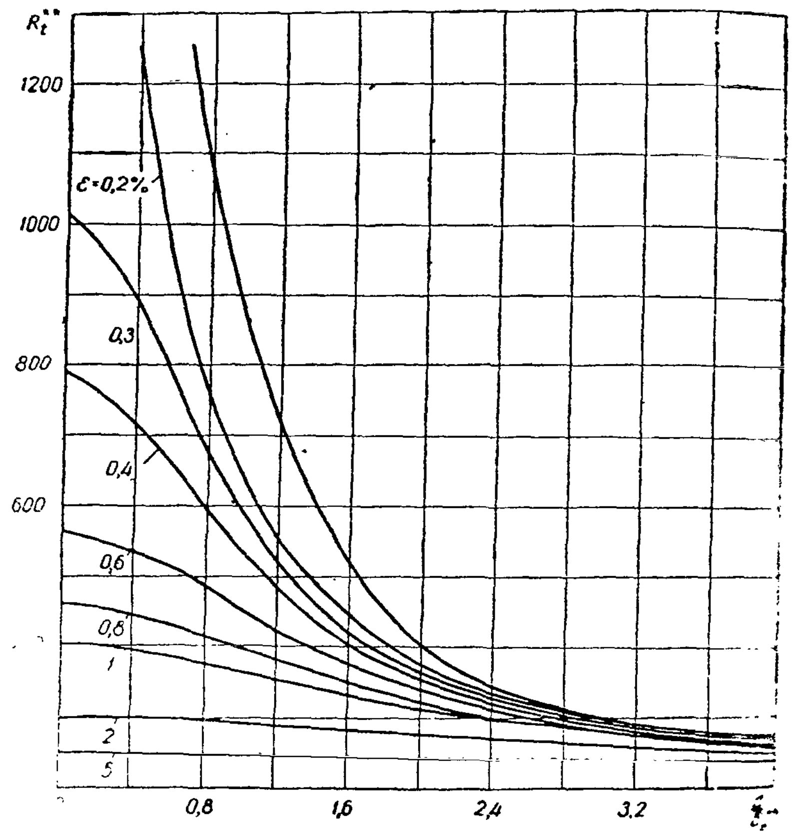


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Fig. 61. The Critical Reynolds Number for a Plate as a Function of the Level of Initial Turbulence and the Roughness Parameter.

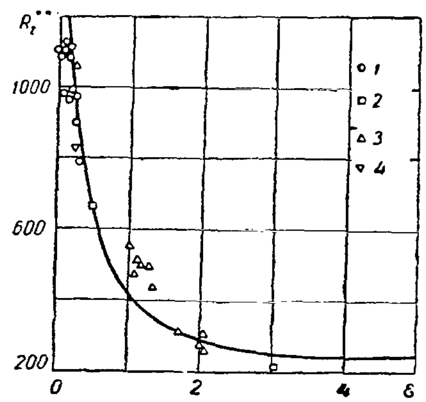


Fig. 62. Comparison of the Experimental Values of the Critical Reynolds Number with the Calculated Values: —— the Calculation ccording to Formula (6.38); (1) According to the Experiments of [105]; (2) According to the Experiments of [59]; (3) According to the Experiments of [71]; (4) According to the Experiments of [126].

Knowing the critical Reynolds number at the transition point, we can compute from equation (6.24) the optimal distribution of the rate of suction from the boundary layer, and then by any known method we can compute all the remaining characteristics of a boundary layer and the profile drag of the body.

To satisfy the condition that the local Reynolds number be equal to its critical value at the transition point, computations

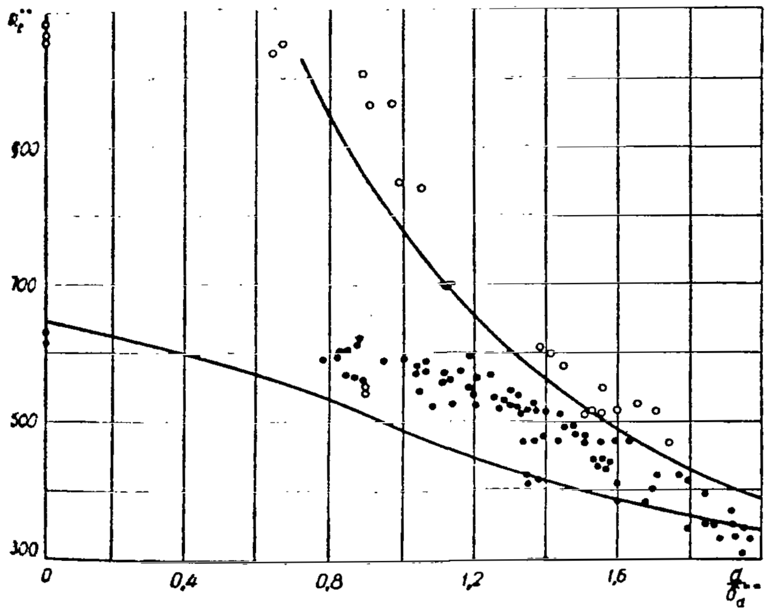


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Fig. 63. Comparison of Calculated and Experimental Values of the Local Critical Reynolds Numbers at the Transition Point: —— the Calculations According to Formula (6.35); ○ - According to the Experiments of [108]; ● - According to the Experiments of Fiendt [56].

should be made by the method of successive approximations. In the first approximation we should ignore the longitudinal pressure gradient on the outer boundary of the boundary layer and determine by formula (6.24) the distribution of the optimal suction rate along the contour of the body. In the second approximation the values of $R^{**}_t$ and $\frac{d}{\delta^{**}_d}$ become more precise, allowing for the law of fluid suction determined in the first approximation. Repeating the process of successive approximations, we can compute with the required accuracy the distribution of the optimal suction rate and also all the remaining characteristics of a boundary layer and the profile drag of the body. Practical calculations showed that for a body with a ratio of length to greatest width of more than seven, the second and third approximations are practically the same. /178

## Optimal Suction of the Boundary Layer of a Plate

For the particular case of a porous plate, expressions (6.21) and (6.24) can be reduced to an integral expontential function, detailed tables of which are presented in reference [18]. Later, when examining the particular case of a plate we will use a rectangular system of coordinates. The origin will be placed at the leading edge of the plate, the $x$ axis directed along the surface of the plate and the $y$ axis normal to that surface. The equation of the zero-th moment (6.14) in this case can be written in the form of the integral relationship /179

$$\frac{d}{dx}\int_0^{\infty} u(U-u)\,dy + v_0 U = \frac{\tau_0}{\varrho}, \qquad (6.41)$$

where $U$ is the velocity of the leading flow; $v_0$ is the local suction rate; $\tau_0$ is the friction stress on the surface of the plate; $\rho$ is the density of the fluid.

We will use the following values:

$R^{**} = \frac{U\delta^{**}}{\nu}$ and $R_x = \frac{U_x}{\nu}$ local Reynolds numbers

$t^{**} = \frac{v_0\delta^{**}}{\nu}$ suction parameter

$\zeta = \frac{\tau_0\delta^{**}}{\nu\varrho U}$ dimensionless coefficient of local friction

where $\delta^{**}$ is the thickness of the impulse loss of the boundary layer, $\nu$ is the kinematic modulus of fluid viscosity.

Then we convert relationship (6.41) to the form

$$R^{**}\frac{dR^{**}}{dR_x} + t^{**} = \zeta. \qquad (6.42)$$

When calculating the lowest critical Reynolds numbers by the method of small perturbations, scientists usually investigate the stability of the fluid flow in the laminar boundary layer. This method of investigation is very time consuming. We can take advantage of the fact that the influence of various laws of suction rate distribution along the plate on the loss of stability of the flow in a laminar boundary layer can be brought into line by using the shape factor $H$. The value of the lowest critical Reynolds number can be computed using the approximate formula (6.3) in which for the case of a plate we should take $A$ = 31.3 and $B$ = 10.

Using a system of equations in three moments with an arbitrary distribution of the suction rate along a porous plate for $\zeta$ and $H$, we obtain the approximate formulas /180

$$\zeta = \zeta_0 - dt^{**}; \qquad (6.43)$$

$$H = H_0 + ct^{**}, \qquad (6.44)$$

where $\zeta_0$ = 0.22; $d$ = 0.56; $H_0$ = 2.59 and $c$ = 1.18 (constant values) and $t^{**} \leq 0$.

Comparison of the results of calculations according to formulas (6.43) - (6.44) with the data of numerical integration of the differential equations for a laminar boundary layer on calculating machines [16] showed that with $t^{**}$ = 0-0.5, the maximum error in formula (6.43) does not exceed about 3% and in formula (6.44) it does not exceed 1%.

Substituting formulas (6.43) - (6.44) into equation (6.42) and separating the variables, we obtain the ordinary differential equation

$$(-bc)\frac{\exp\left[2\left(a-H_0b-bct^{**}\right)\right]}{\zeta_0-(d+1)t^{**}}dt^{**}=dR_x. \qquad (6.45)$$

Integrating equation (6.45) with the help of the change of variable $z=\frac{2bc\zeta_0}{d+1}-2bct^{**}$ with the boundary conditions $t^{**}$ = 0, $R_x$ = $R_{x0}$, we obtain

$$R_x-R_{x_0}=\frac{bc}{d+1}\exp\left[2\left(a-H_0b-\frac{bc\zeta_0}{d+1}\right)\right]\times$$
$$\times\left\{E_i\left(\frac{2bc\zeta_0}{d+1}-2bct^{**}\right)-E_i\left(\frac{2bc\zeta_0}{d+1}\right)\right\}, \qquad (6.46)$$

where $E_i$ is an integral exponential function[5].

Since [99] $R_H^{**}$ = 225, it is easy to obtain that $R_{x0}$ = $0.115\cdot 10^6$. After computing the function $R_x(t^{**})$, using the graphic function $R^{**}/\sqrt{R_x}$ from $t^{**}$ [16], we can determine the coefficient of local discharge,

$$\frac{v_0}{U}=\frac{t^{**}}{R^{**}}. \qquad (6.47)$$

Figure 64 shows a comparision of the functions obtained by different authors for the rate of optimal suction on a porous plate $v_0/U_0$ with various values of the Reynolds number $R_x = \frac{U_0x}{\nu}$, where $U_0$ is the velocity of the leading flow. Comparison of the curves /181

[5] Detailed tables of the integral exponential functions are given in reference [8], these tables are recommended for use in practical calculations.

shows a satisfactory correlation of the results of calculations according to the proposed formula with the data from reference [125].

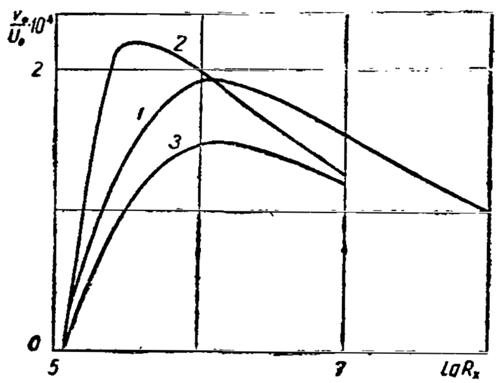


Fig. 64. The Optimal Suction of a Boundary Layer of a Porous Plate as a Function of the Reynold Number: (1) According to [18]; (2) According to [123]; (3) According to [125].

The drag coefficient of laminar friction with optimal suction of a boundary layer on a porous plate can be computed by the formula

$$\zeta_f = \frac{2\int\limits_0^x \tau_0 dx}{\varrho U_0^2 x} = \frac{2}{R_x}\int\limits_0^{R_x} \frac{\zeta \frac{v_0}{U_0}}{t^{**}} dR_x, \qquad (6.48)$$

which with the help of expression (6.44) can be converted to the form

$$\zeta_f = \frac{2\xi_0}{R_x}\int\limits_0^{R_x} \frac{dR_x}{R^{**}} - \frac{2d}{R_x}\int\limits_0^{R_x} \frac{v_0}{U_0} dR_x. \qquad (6.49)$$

Taking into account that for $R_x \leq$ /182 $R_{x0}$; $\frac{v_0}{U_0} = 0$ and $R^{**} = 0.664\sqrt{R_x}$, in the final form we obtain

$$\zeta_f = \frac{1{,}328\sqrt{R_{x_0}}}{R_x} + \frac{0{,}44}{R_x}\int\limits_{R_{x_0}}^{R_x} \frac{dR_x}{R^{**}} - \frac{1{,}12}{R_x}\int\limits_{R_{x_0}}^{R_x} \frac{v_0}{U_0} dR_x. \qquad (6.50)$$

Both the integrals in formula (6.50) were determined by graphic integration using function (6.46). In this case, we considered that $\frac{v_0}{U_0} \leq 0$.

Table 13 shows the results of calculations according to formula (6.50) of the drag coefficient of laminar friction with optimal suction of the boundary layer.

For practical calculations it is necessary to compute the coefficient of total discharge, determined by the formula

$$C_v = \frac{Q}{U_0 x} = \frac{1}{R_x}\int\limits_{R_{x_0}}^{R_x} \frac{v_0}{U_0} dR_x. \qquad (6.51)$$

TABLE 13. THE DRAG COEFFICIENTS AND THE DISCHARGE OF SUCKED-OFF FLUID AS A FUNCTION OF THE LOCAL REYNOLDS NUMBER FOR THE CASE OF A PLATE.

| $R_x \cdot 10^{-6}$ | $\zeta_{f_0} \cdot 10^3$ $\left(\frac{v_0}{U}\right)_{opt}$ according to (6.51) | $\zeta_{f_c} \cdot 10^3$ $\left(\frac{v_0}{U_0}\right) = 1{,}18 \cdot 10^{-4}$ according to [100] | $\zeta_{f_c} \cdot 10^3$ $\left(\frac{v_0}{U_0}\right) = 2{,}0 \cdot 10^{-4}$ according to [100] | $\zeta_{fl} \cdot 10^3$ according to Blasius | $\zeta_{ft} \cdot 10^3$ according to Prandtl+ Schlichting | $\frac{\Delta\zeta_l}{\zeta_{ft}}$, % | $c_Q \cdot 10^3$ |
|---|---|---|---|---|---|---|---|
| 0,115 | 3,920 | 4,220 | 4,360 | 3,920 | 7,100 | 0 | 0 |
| 1,00 | 1,338 | 1,471 | 1,680 | 1,328 | 4,450 | 70 | 0,216 |
| 10,0 | 0,683 | 0,590 | 0,736 | 0,421 | 3,100 | 78 | 1,552 |
| 100 | 0,340 | 0,332 | 0,450 | 0,133 | 2,131 | 84 | 1,206 |

We will now examine in more detail the particular case of optimal suction of a boundary layer on a plate taking into account initial turbulence and surface roughness. In this case the basic calculating formulas are significantly simplified and take the form [25] /183

$$R_t^{**} = \exp(A - BH) + \frac{C_1}{\varepsilon^{\frac{5}{4}} + D\left(\frac{d}{\delta_d^{**}}\right)^2}\left[f_{\varepsilon_0} - \frac{k}{a}(H - H_0)\right]^{\frac{1}{2}}; \qquad (6.52)$$

$$R^{**2} = \left[2\left(\zeta_0 + \frac{d+1}{c}H_0\right) - \frac{2(d+1)}{c}H\right](R_x - R_{x_0}) + R_0^{**2}; \qquad (6.53)$$

$$\frac{v_0}{U} = \frac{\nu}{U}\cdot\frac{1}{B-2}\cdot\frac{1}{R^{**}}\cdot\frac{dR^{**2}}{dx} - \frac{a}{B-2}\cdot\frac{1}{R^{**}}. \qquad (6.54)$$

These formulas allow us to carry out the necessary computations. Figure 65-67 show the optimal distribution of the rate of suction of the fluid from the boundary layer of a plate as a function of the Reynolds number $R_x$ for various values of the initial turbulence of the leading flow and surface roughness. From an analysis of the data shown in these graphs, it follows that the optimal distribution of the rate of suction of the fluid from the boundary layer of a porous plate and the total discharge of sucked-off fluid vary to a significant degree with the initial turbulence and the surface roughness of the body.

With low initial turbulence of the flow ($\varepsilon$ = 0.2%) (Fig. 65) the value of the surface roughness exerts a significant influence on the optimal suction. In this case the required quantity of sucked-off fluid increases in proportion to the increase in roughness. With significant initial turbulence ($\varepsilon$ = 2%) (Fig. 66) variation in the surface roughness does not exert a significant influence on the optimal suction of the boundary layer.

The distribution of the rate of suction of the fluid from the boundary layer of a permeable plate as a function of the Reynolds number $R_x$ and the initial turbulence for the case of an absolutely smooth surface ($d = 0$) is shown in Figure 67 (the broken line shows the data of Pretsch). It follows from Figure 67 that the optimal distribution of the rate of suction of the fluid from the boundary layer of a permeable plate and the total discharge of sucked-off fluid vary to a significant degree with the initial turbulence. Thus for example, the maximum value of the dimensionless rate of optimal suction of the fluid with an initial turbulence $\varepsilon = 2\%$ is $\frac{v_0}{U_0} = 1.24 \cdot 10^{-4}$, and with an initial turbulence $\varepsilon = 0.2\%$ the corresponding value is equal to $0.56 \cdot 10^{-4}$. It is natural that as the initial turbulence decreases, the required discharge of fluid sucked from the boundary layer across the permeable surface of a plate decreases.

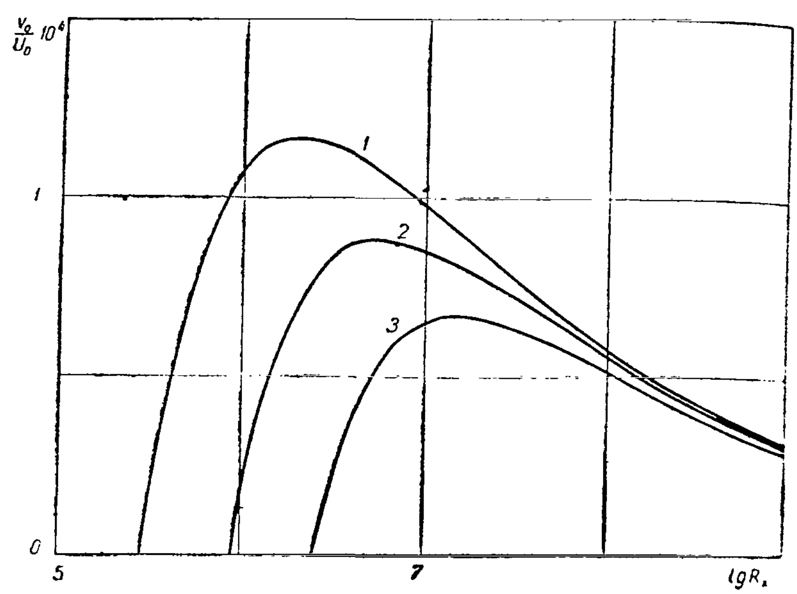


Fig. 65. Optimal Distribution of the Suction Rate for a Plate with Various Surface Roughnesses (1) $\frac{d}{\delta_d^{**}} = 4$; (2) $\frac{d}{\delta_d^{**}} = 2$; (3) $\frac{d}{\delta_d^{**}} = 1$.

The numerical values of the coefficients of friction drag for a plate with optimal suction of a laminar boundary layer as a function of the Reynolds number $R_x$ with various initial turbulences and surface roughnesses are shown in Figure 68. From these data if follows that laminarization of the boundary layer of a plate by suction of a fluid across the permeable surface allows us to decrease significantly the hydrodynamic resistance.

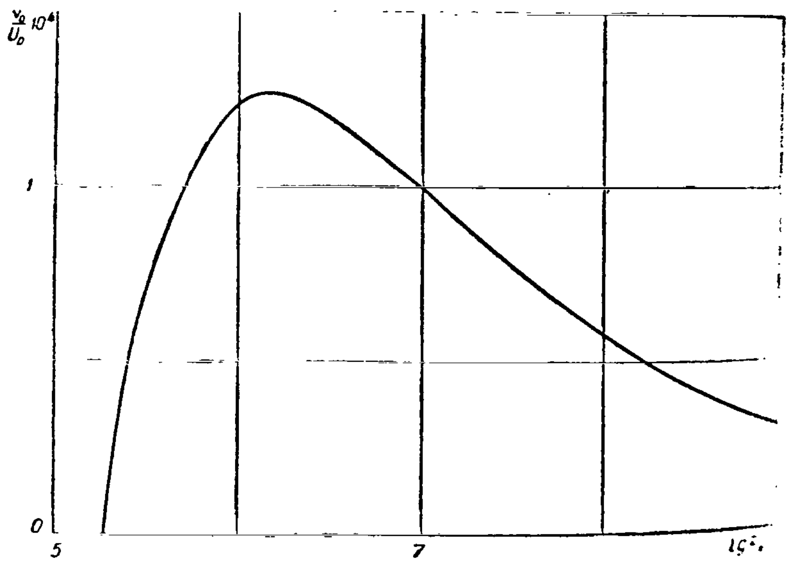


Fig. 66. Optimal Distribution of the Suction Rate for a Plate with Various Surface Roughnesses.

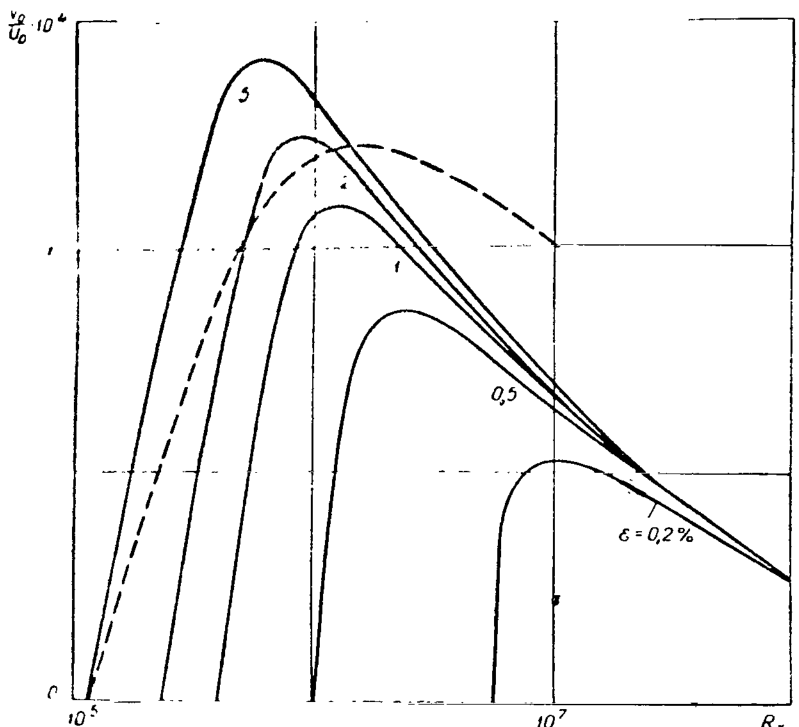


Fig. 67. Optimal Distribution of the Suction Rate for a Plate with a Smooth Surface with Various Initial Turbulences in the Flow.

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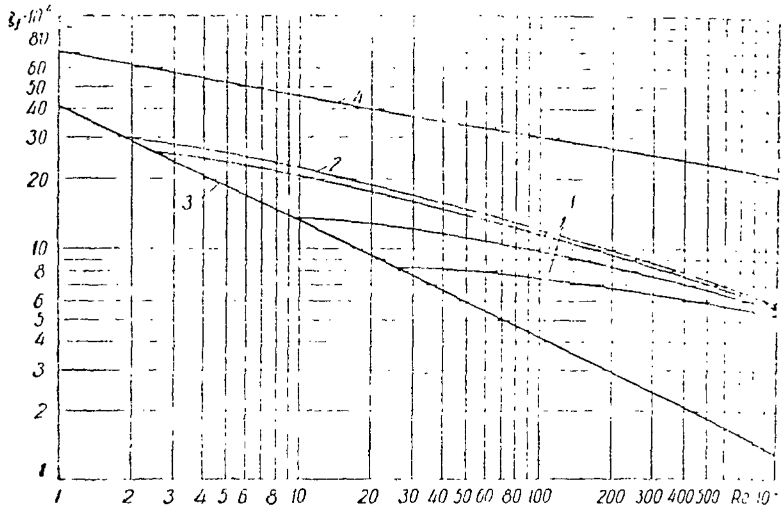


Fig. 68. The Coefficient of Friction Resistance as a Function of the Reynolds Number for Various Surfaces Roughnesses and Initial Turbulences of the Flow: (1) Optimal Suction with ε = 0.2%; (2) the Same with ε = 2% (3) the Laminar Boundary Layer According to Blasius; (4) A Turbulent Boundary Layer According to Prandtl-Schlichting.

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